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THE

MATHEMATICAL GAZETTE.

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NOTES ON THE TEACHING OF ARITHMETIC.

BY MRS. F. G. SHINN.

THE following notes are from a paper by Mrs. F. G. Shinn, read to the London Branch, October 24, 1914, and entitled "Some Points in the Teaching of Elementary Arithmetic."

SUBTRACTION.—I do not advise the teaching of subtraction by complementary addition. To the child it adds only another difficulty to the already difficult subtraction, but it will come in naturally later, if, before beginning any subtraction, the children have a thorough knowledge of the composition of numbers, and have a great deal of practice in quick mental additions, such as 4 and 3, 5 and 4, etc. And here note that though the parts of numbers up to 10 are really sufficient, others should also be taken, because to the child's mind 15 and 3 is far more difficult than 5 and 3. Good practice in this is got by taking the number 1 and adding 3 to it, and successively to the sums, quickly round the class, 1, 4, 7, 10, etc., and doing the same with the other numbers up to 9. Then reverse the process by subtraction, starting at 50 or 100. You will probably find that the combination of certain numbers always presents difficulties to the same child, and practice in these special ones should be given.

I always add on to the bottom line, and my reasons are, (i) when pupils taught in this way come to decimals and long lines of noughts, they experience no difficulty, and (ii) without it, Italian Division is impossible. If you have the sum 73 − 35, say :—"5 from 3 you cannot" (we *have* to say this), "5 from 13 leaves 8"; then you have already used one of the next and you have to take 3 more, so say :—"4 from 7 leaves 3"; answer 38. This can be best illustrated by money in a purse consisting of shillings and pence. If you have to pay a small bill and have not pennies enough, you take, *not borrow*, one shilling and call it, or turn it into 12 pennies, and then can pay the required amount.

How far are explanations desirable at this early stage? The old method of teaching was to give a rule and set numerous examples on it—the longer the better—till the pupils were quite proficient, and it was almost impossible for them to make mistakes. Now, I am not sure that having left this obviously bad method, we have not gone too far in the other direction, and labour over long explanations of everything, which only confuse the pupils. The memory for tables, and the faculty of manipulation of figures advance much faster at this

stage than the power to comprehend reasoning, and therefore I think explanations are best left alone at an age when they are not really intelligible. Always have a reason for everything, in case an enquiring child asks for it, and at this stage use plenty of concrete illustrations, but leave abstract reasoning alone until later.

I think it very essential that horizontal subtractions and additions should be taught. Children have a way of thinking that they cannot do a sum set down in unfamiliar guise. In practice, horizontal subtraction occurs most frequently. Exercises might also be given in subtracting the top line from the bottom, but not, of course, until the other is thoroughly known, or confusion will arise. A good many sums should be given out in words, *e.g.* "three thousand and eighteen from ten thousand," or "add together seventeen, a hundred and twenty, three thousand and ninety-six, twenty-seven, and five."

ITALIAN DIVISION saves space, as only half the figures are written down, but I am doubtful whether it is shorter in point of time; and if the quotient contains any number repeated, there is a distinct disadvantage, as the multiplication has to be repeated each time.

Science masters or mistresses often complain that the pupils have no idea of area or long measure, and are ignorant of decimals. This is often said of girls from fourteen onwards, who may be getting on well with, say Interest or Stocks—far beyond the points mentioned. And these complaints are often only too true, as a glance at the Science note-book will show.

Students may be "very good at Arithmetic," and yet have no real conception as to dimensions, and may be unable to apply their knowledge. How is this to be remedied?

DECIMALS.—Unfortunately a large number of incompetent people have to teach decimals. But this does not account for it entirely—I have had pupils to whom I have taught decimals from the beginning, who rarely make mistakes as to the point in their Arithmetic lessons, but of whom the Science mistress has complained that they never get a point in the right place in working out their Science results. They have been told so often in the Arithmetic lesson that method is everything, and they necessarily think of it, almost unconsciously—but in the Science work the result is the main thing. I think part of the mischief is begun in the very first lesson we give on decimals. We explain what decimals are, how important position is, etc. We then take any number which comes into our heads, say 34258976; we put the point in anywhere, say after the 2, and make somebody read it out as "three hundred and forty-two point 58976," and somebody else explain it in tens, tenths, etc. We then move the point somewhere else and go through the same process many times, until we think the whole class understands thoroughly. But may not this be at the root of some of the trouble? They see us taking out points and putting them in anywhere at will, and the child's illogical mind thinks that the position of the point does not matter. I don't know whether this may be so, but suggest that a better method would be to use different figures each time, or to let a pupil make up a number about 10 or 100 times as large as the one we have chosen and explain it.

Before I leave the subject of decimals, I should like to say a word about *Multiplication*. I think there is something good in every method, but to my mind the easiest and the best method in most cases is to put the unit of the multiplier under the last figure of the multiplicand and begin each partial product under the figure of the multiplier used, the point keeping the position it has in the multiplicand. This method is also very simple to explain. I do not think it matters at which end of the multiplier you begin. I know the argument is: Begin with

the left-hand figure, because it is the most important; but I do not think much of that, because we do not always begin with important things, but more often with unimportant and lead up to the others. The argument in favour of beginning at the right is that it is most natural, and the multiplying *must* begin at the right of the multiplicand. I once found a child who had been told to begin with the left-hand figure also trying to do the multiplying from that end and carrying backwards. One thing against this method of multiplying decimals is that the position is sometimes awkward, as in $45{\cdot}321 \times {\cdot}00425$, but then why not make 45·321 the multiplier, except that we should have 5 lines of multiplication instead of 3? Another way of avoiding the difficulty is to do much the same as we do in division and multiply by 4·25, and then divide the answer by 1000—this is hard, but it is good practice in the position of the decimal point.

The only thing in favour of the method of putting the point under the point is the neat look of the sum. Getting the position of the figures for the first line of multiplication is hard and needs thought, but the other lines fall into their places mechanically when the first is done. For this reason I should always start multiplying by the unit of the multiplier.

Most of us were taught to multiply the figures, and to put in the point last in the answer by adding the number of places in the two original decimals. This, I believe, is considered with horror nowadays, and is never allowed. However, I think there is something good even in this method, though I should not myself teach it. After all, it is only really using the principle of indices, and also has the merit of always getting the sums right.

I think it is a good thing to introduce at a very early stage the idea of giving decimal answers correct to so many places. It is always required in any practical work, and if a child makes a slip at the beginning of a long division sum, she will not waste time wandering on for pages in the hope that it will come out, but will stop after 3 or 4 figures and give the answer "correct to" one less. But if some other term than "correct to" could be found it would be better. Do not let children leave off and give the answer as far as found, and never allow such an answer as 3·425 approximately, but make them state how near, as, for instance, 3·425 correct to 3 places.

LENGTHS, AREAS, VOLUMES.—We have continually to correct such statements as:—"*Area* of floor = so many *yards*," or "*volume* of tank = so many *sq. ft.*" Or if we give as two consecutive sums to fill a tank and to paint the inside of a cistern, we are sure to have some who fill the cistern with paint by taking a volume, or put a layer of water all over the tank by taking areas, according as to which sum is given first. In most cases this is a sign that the idea of dimensions has never been really understood. Now this is very difficult to remedy later, and we must simply hammer in these ideas continually at the very beginning of this part of the work and set numerous catchy questions, and never pass over lightly a fault of the kind mentioned above.

Children should be made to have a more definite idea of the lengths and weights they are using. Have a yard- and a foot-rule and a lb. weight in the schoolroom; let them know that a halfpenny measures an inch across, and things of that sort. Try the experiment of telling the class to hold up their hands a yard apart, to draw a line 4 inches long, to guess the length of the room, or the weight of a book, etc. Rough estimates of what the answer will be should, where possible, be always given, and thus many absurd mistakes will be avoided.

PROBLEMS.—Most text-books introduce their chapters on problems too late and at too infrequent intervals. The child, in consequence,

thinks of problems as necessary evils, which turn up occasionally in sets. I think that some problems should be sprinkled in every set of examples—begin with as easy ones as you like, such as "I have 6d. to spend; I go to a shop which sells penny pencils; how many can I buy? If the pencils are 2d. each, how many? If I have 10d. and the pencils are 4d. each, how many can I afford, and how much change shall I have?" This sort of thing will form the habit of thinking out problems, and the teacher must insist upon statements of even the simplest ones being written down—thinking on paper I call it.

These easy examples will, of course, soon lead up to harder ones, and introduce x as early as you like—let it come in naturally without first going into the question of Algebra. In an example, for instance, about the distance between places: suppose we know all about it, and make up statements about the rest of the example, calling it "the distance" each time, then say "Now for shortness, to save writing the words 'the distance' so often, write x instead."

EXAMPLES.—Of course some on certain rules must be put in sets in a text-book, otherwise we should never know where to find anything; but the inexperienced teacher allows a class to work through each set in turn, and thinks that when this has been done the class knows all about the rules illustrated. But this may not be so at all. A certain amount of mechanical proficiency is acquired, but after the first sum in each set little or no thinking may be required.

To avoid mechanical working of sets I adopted, for class work, the plan of working, say, four sets at once, the first in each, then second in each, and so on. The chances were that no two sums alike were worked on the same page, and, of course, no looking back was allowed. For homework, it is sometimes a good plan to dictate sums from a private book of your own. The form of the question will be slightly different from their text-book, and also they will not know without thought what rule the sums illustrate.

I do not intend to speak about higher work—Interest, Stocks, Profit and Loss, etc.—but I should like to say what I consider the best way of dealing with sums of money which come into all these as well as into the earlier work. For example: find the value of $\frac{8}{15}$ of £19. 3s. 9d. Some people would say divide by 5 and 3, and multiply the result by 8, keeping in pounds, shillings and pence all the time. This may be very easy in some cases, but the following method is possible in all: work in fractions of a pound by the help of shillings if necessary. By that I mean—all sums having 6s. 8d., 13s. 4d., 7s. 6d., and other well-known parts of a pound can be done in pounds at once. But most other examples can be done more easily in this way. We have to take two steps to get from pence to pounds, so take one in each direction from pounds to shillings and from pence to shillings, as both these, being independent, can be done at once, and put a 20 underneath to get back to pounds—always work in pounds and not shillings, because of getting the result correct to 3 places and reading off by inspection. To return to the example given:—

$$\frac{8}{15} \text{ of £19. 3s. 9d.}$$

$$= \frac{8}{15} \text{ of £} \frac{383\frac{3}{4}}{20}$$

$$= \text{£} \frac{\overset{2}{\cancel{8}}}{\underset{3}{\cancel{15}}} \times \frac{\overset{307}{\cancel{1535}}}{\underset{1}{\cancel{4}} \times \underset{10}{\cancel{20}}}$$

$$= \text{£}10{\cdot}2333$$

$$= \text{£}10.\ 4\text{s.}\ 8\text{d.}$$

This example could have been worked in many ways, some shorter than mine, but I have chosen it merely to show this method. Let us take another instance :—

$$\frac{40}{263}\times\frac{16}{100}\text{ of £58. 14s. 7}\tfrac{1}{2}\text{d.}$$

$$=\frac{40}{263}\times\frac{16}{100}\text{ of £}\frac{1174\frac{1}{2}}{20}$$

$$=\text{£}\frac{\overset{2}{\cancel{40}}}{263}\times\frac{\overset{2}{\cancel{16}}}{\underset{5}{\cancel{100}}}\times\frac{9397}{\underset{10}{\cancel{2}}\times\underset{5}{\cancel{20}}}\quad\text{(Do not cancel 4 into 100.)}$$

$$=\text{£}\frac{375{\cdot}88}{263}$$

$$=\text{£}1{\cdot}429$$

$$=\text{£}1.\ 8\text{s.}\ 7\text{d.}$$

$$263\overline{)375{\cdot}88(1{\cdot}4292}$$
$$\underline{263}$$
$$1128$$
$$\underline{1052}$$
$$768$$
$$\underline{526}$$
$$2420$$
$$\underline{2367}$$
$$53$$

No rough work is required except in the final long division. These types are always turning up, and the way of working them out presents many difficulties as to method and arrangement, but I think the above is the best, as it is useful in all cases, and therefore a good one to teach. Sometimes, of course, there is a small piece of rough work required earlier, as in £172. 12s. 3½d.; we get $\frac{£3452\frac{7}{24}}{20}$, and the multiplying by 24 cannot be avoided. But this consists in putting down only one line of figures, *e.g.* 13808, which is 4×3452, and the result of multiplying this by 6 can be put down at once in its correct place with the 7 added in.

Corrections ; Rough Work ; Marking.—You will agree that the way a sum is worked is much more important than getting the correct answer, and therefore I should, as a rule, mark a sum with the wrong answer worked in the right method and with clear statements more highly than one with the right answer done anyhow. But if you adhere strictly to this rule, you will find the class will in time become very inaccurate, and you will have, perhaps, beautifully arranged work continually spoilt by slips. I do not believe in being consistent in marking Arithmetic, because surprises are good for children, and also we have so many pitfalls to avoid. For instance, to take the opposite extreme to that mentioned above. Suppose you find your class becoming inaccurate, and so determine to be down on careless mistakes. If you keep this up too long, you will perhaps have an inaccurate child who frequently loses many marks for slips, who will say to herself "I always try so hard to do my Arithmetic nicely and yet I never seem to get any marks." And soon such a child will get disheartened and give up trying to do it nicely. But if on some days you are down on slips and give little credit for a sum with the wrong answer, and on others talk about "only a slip" and give nearly full marks, you will avoid these two extremes, and if you are consistently inconsistent in this way, it will work out fairly for the class in the long run.

Then as to corrections—there are some people who go on the lines that no sum should be left until the pupil has got it right, and therefore you find corrections and re-corrections and even re-re-corrections. The theory of this is right, but the practice is wrong. So much time spent over one sum could so much more profitably be given to new work. And if corrections are done at home before the new homework is attempted, it may mean that so much time is spent over the old work that the new is scamped through in the little remaining time. I believe much more in giving home corrections only occasionally, but in more

frequently going through the homework in class and allowing those who have it wrong to correct from the board there and then. If you find that the same girls make the same mistakes time after time, it means that they correct mechanically from the board, and other measures must be adopted in those cases.

I am very much against so-called rough work done roughly, or on any scrap of paper—firstly, because if any calculation required can be written down anywhere, *no* mental work will be done at all, while if all such work has to be shown up neatly, they learn to do without much of it, and in time get quite good at mental calculations. Secondly, if rough work is done untidily, figures are wrongly read by the pupils—a badly made six, for instance, read as a nought. Thirdly, the scraps of paper get lost, and if the rough work is not shown with the sum it adds very much to the labours of the teacher who is trying to detect a mistake. Fourthly, the sum may have quite the wrong answer, but the rough work may show on what lines the pupil's mind was running, and these may be more or less correct. And, lastly, it is good to form the habit of tidy work, as this tells so much in the candidate's favour in an examination. Therefore insist that every piece of calculation required that cannot find its place in the sum itself should be shown neatly on the same page as the sum, either in a margin at the right, or, if there is no room there, then have it at the bottom of the page.

We must also aim at quickness, for quick work in Arithmetic is not necessarily either hasty or superficial work. Slowness and deliberation in Arithmetic are not good, and increase if not checked.

It is no use, as a rule, *stopping* to think. Concentrated attention is what is required. In order to acquire quickness and accuracy it is a good plan occasionally to set a large number of very easy sums, and to let the pupils work against time, or to divide them into two groups and have a match. In order to make pupils concentrate, it is well occasionally to set long sums, but on the whole these are to be avoided, as two shorter sums have more educational value. H. K. SHINN.

NOTES ON THE BOARD OF EDUCATION CIRCULAR, No. 851.

ON THE TEACHING OF GEOMETRY.

By Miss M. J. PARKER.

CIRCULAR No. 851 does not differ materially from that of 1909, in fact it states that "subsequent observation and discussion have only confirmed the Board's Officers in the views then expressed, and these have on the whole met with considerable acceptance. The main practical suggestion has been adopted by the Committee of the Mathematical Association, and by them pressed upon the Cambridge Syndicate and the Oxford Delegacy, hence the alterations which we now see in the Oxford and Cambridge regulations," and I might add also in the Civil Service examination, but I believe not in those of the University of London. This circular states the matter afresh, explains it more fully, and justifies the attitude adopted.

The main doctrine is that certain propositions which had hitherto come at the beginning of deductive Geometry should be taken for granted as a direct outcome of intuition, and that subsequent structure should be based upon them instead of on the ultimate postulates to which they could be logically reduced. The following list is that given

as all that are necessary at the outset, though one should be added in dealing with proportion and possibly some (not stated) for circles. (I should be glad to know what those are.) (1) If one straight line stands on another straight line, the adjacent ∠s are together equal to 2 rt. ∠s, and (2) its converse; (3) vertically opposite angles are equal. (4) If one straight line cuts 2 st. lines so as to make the alternate ∠s equal, the exterior equal to the interior opp. ∠s equal or the 2 int. ∠s together equal to 2 rt. ∠s, the straight lines are parallel, and (5) its converse; (6) the three angles of a △ are together equal to 2 rt. ∠s; (7) the three theorems on congruence of △s, *i.e.* three sides, two sides and included angle, two angles and corresponding side; (8) angles at the base of an isosceles △ and its converse.

These are to be treated with absolute freedom; teachers should use whatever method they find best—intuition, observation, experiment, or, even when they find it acceptable, deduction. What is essential is that the propositions should be firmly and completely grasped, and believed. They are the tools with which the boy is to work. Later on, when he has made real progress in Geometry, it may be quite advisable to return to the beginning and examine such statements as have previously been accepted as fundamental after the "boys" have mastered at least the whole contents of Euc. I.-IV. and VI., and some solid Geometry. The circular goes on to justify this suggestion, showing that the logical accuracy of the deduction may be equally sound whether it starts from assumptions which the philosopher finds to be ultimate, or from those which the teacher may adopt as suitable for the beginner. It goes into the matter in detail, discusses the difference between the philosopher and the school boy, considers the use of definitions, and then goes on to say that when once the fundamental beliefs or assumptions have been made the secure possession of the pupils, the character of the subject changes; henceforward the process is that of Euclid, only it commences at a different point and one more suitable for beginners.

It states (Art. 68) that not even the slightest course of School Geometry can be considered reasonably complete unless it includes the elements of proportion and similarity, stating that all examining bodies, with one prominent exception (I take it that the exception is the University of London), include this section in their definition of Elementary Geometry. It suggests that this may be taken at any time after the establishment of the proposition that if 3 parallels cut off equal segments from one transversal they do from all; that whenever this subject of ratios and similar figures is introduced it is desirable, unless it has been done earlier, to make pupils familiar with the trigonometrical ratios, at least the sine, cosine and tangent; and states that it is now becoming widely recognised that simple numerical trigonometry should form an early part of a mathematical course.

It deals with the use of exact drawing, which it states is of most importance when new ideas, such as envelope and locus, have to be assimilated, and should not be restricted to examples which can be proved theoretically by the means at the pupils' disposal. It suggests that propositions should be dealt with in groups, not separately, and ends with such practical questions as the marking of written work, and a discussion on blackboard work, to which I will refer later.

Now with regard to the main doctrine. It would be interesting to know how many schools represented here have adopted this plan, omitting the proofs of congruence of △s and the theorems on parallels at the beginning of deductive Geometry, and at what stage they have taught them; for up to the present, for the requirements of examinations, they have to be taught before the circular considers it advisable.

At my own school we adopted the plan to some extent some five or six years ago, and I must say I think it has answered, though one of my colleagues found it a relief to go back to the deductive proofs. After a year of what we call practical Geometry, or an introduction to Science and Geometry, the girls began deductive Geometry with circles (we find circles appeal to them), but proved these fundamental theorems before the end of their first year of deductive Geometry. We did it with a view to making the beginning easier, and it certainly answered, but we found the girls were quite ready for these proofs, and took to them before the end of the year. Now we are starting the plan of leaving them until considerably later; perhaps by the time these girls reach the stage of public examinations, the proofs will not be required at all.

But to come to details. If we are to gather by intuition that 2 $\triangle$s are congruent if 3 sides of the one = 3 sides, and 2 $\angle$s and a corresponding side = 2 $\angle$s and corresponding side, or 2 sides and included $\angle$ = 2 sides and included $\angle$, and we are also to take propositions in groups, why not also take as intuitive the case when we have the hypotenuse, a right $\angle$, and one side? If we do so we can begin deductive Geometry with the theorems on chords of a circle, and other theorems on $\odot$s, something quite fresh and interesting to the pupil, because there is something definite to prove.

With regard to $\parallel$ lines, I find that the girls see at once that the exterior angle equals the interior opposite angle (by the way I like the term corresponding $\angle$s, as the girls get confused with ext. and int. opp. $\angle$s of $\triangle$). The fact can be illustrated by the sliding of set squares, by rotation, by two men walking along a road, one turning off and the other walking on a little farther and then turning off into a road $\parallel$ to the other man's; and I think this is the fundamental assumption upon which we should base future facts. To illustrate the fact that alternate $\angle$s are equal, I personally do not find as easy. The man walking along the broken line *AXYD*, as referred to in paragraph 48, I do not think at all satisfactory, for my idea is that the man does not turn through the alternate angles at all. It is true the girls see at once that alternate $\angle$s are equal, but I think they see it because vertically opposite $\angle$s are equal; in fact they use a deductive proof, a good introduction to the deductive proofs that are to follow. The sum of the two interior angles together equal to two right angles will also follow deductively from adjacent $\angle$s being together equal to 2 rt. $\angle$s.

I do not think that we should avoid the deductive proof of 3 $\angle$s of a $\triangle$ together equal to 2 rt. $\angle$s. I quite agree that the girls do not like proving the obvious, but this is not obvious, it is something very definite; the proof is not difficult when once the ideas of parallels have been grasped, and the girls like it. Moreover, to bring it home by intuition is *not* easy. We are warned, and rightly, not to rely on measurement, a most pernicious practice, and one from which even our lecturers in university colleges are suffering; tearing off the corners is equally unsatisfactory; and, as for the man walking along the sides of the triangle, it seems to me he certainly does not turn through those angles but through the exterior angles. The rotation of a pencil mentioned in paragraph 28 is the best, but the fact that you use the different ends of a pencil is to the children somewhat of a difficulty. Of course, as the circular suggests in other cases, we may use such illustrations as the above, but only after the theorem has been proved deductively. Personally, I do not think the child grasps it by intuition at all, and I see no reason why she should be expected to do so. There is a danger that the child will rely on measurement as a sufficient proof of this

theorem: the fact that all the class obtain approximately 180° is satisfactory to her.

We are told that the assumptions should not be unnecessarily numerous (par. 3). We are also told (par. 13) that it is ridiculous to suppose it matters either to a boy's progress in Geometry or to his general soundness of mind whether he deduces the angle sum of a △ from properties of ∥ lines or *vice versa*, or whether he accepts both statements separately as the outcome of observation and intuition. Perhaps not theoretically, but if he deduces angle sum of a △ from parallels in one school and parallels from angle sum in the next, it matters very much, and perhaps that is the reason the Board thinks it is best to take all these intuitively. In paragraph 21 we are told teachers may use whatever methods they find best—intuition, observation, experiment, or even deduction—as long as the propositions are firmly grasped. These propositions are the tools with which the pupil is to work; just how they were made and how they came into his possession matters little, the important thing is that he should have them and learn to use them. Now, of course, all the theorems are tools for future use, though some of them are not used by the ordinary schoolboy. As the circular states in the last paragraph, the whole thing is a structure, and unless the teacher leaves this idea of structure much of his work is valueless. I take it that these fundamental theorems are not exactly to be looked upon as the foundation of the building but the tools with which to build, so that one should not depend on another. Altogether I could have wished the Board had been a little more definite on the subject. The Board, by allowing some of these "tools," as it calls them, to be proved deductively allows the tools to become part of the building. Personally, I should take as few tools as possible and begin the building with the angle sum property of a △, alternate ∠s and then ⊙s.

So that with the main doctrine of the circular, *i.e.* that we should take as fundamental beliefs the facts enumerated in paragraph 17, I should agree, except that I should add one, *i.e.* that 2 rt. angled △s are equal if hypotenuse and one side are equal, and I should omit one, *i.e.* 3 ∠s of a △ together equal 2 rt. ∠s; also in ∥ lines I should take as fundamental ext. = int. opp., and lay far more stress upon this than on alternate ∠s or sum of 2 int. opp. ∠s.

With regard to Proportion and Similarity and Trigonometry (Art. 68, 69), I quite agree. Having taken for granted in our practical work that we may draw to scale, having illustrated from the beginning that two △s are not congruent but of the same shape, if the ∠s are respectively equal, why not make early use of these fundamental and intuitive beliefs and go on to the fresh interest of numerical Trigonometry, including heights and distances, which opens the girls' minds to the facts that Geometry is really used? And in the extension of the Theorem of Pythagoras it is a relief to see we may be allowed to use the word cosine instead of "the rectangle contained by one side and the projection of the other side on it."

At present we learn Geometry for several years, we pass matriculation, we do easy riders which might far more easily be proved by similar △s, and then we are introduced to certain theorems which are perfectly obvious, the truths of which we learnt when we first began to draw, when we made a man's head as big as his body and somehow it did not look right.

With regard to Solid Geometry (paragraphs 70 and 71), I think by this time we must all agree that the ideas of solid figures should be given at the very beginning in the introductory practical course.

Now, with regard to the actual *methods* of teaching, the circular goes into details.

In dealing with congruence of $\triangle$s the circular states in paragraphs 31 and 32 that instead of comparing two ready-made $\triangle$s a more effective method is to draw one $\triangle$ and consider which and how many of its elements must be measured to produce a copy, and goes on to say :— "It is not necessary that the $\triangle$ should actually be constructed from its 3 elements, but only that the process of construction should be carried out in imagination. In fact the sufficiency or insufficiency of the data for a unique determination is probably better appreciated from such imaginative construction than when the free play of thought is hindered by the mechanical difficulties of handling instruments. Such imagination is essential to any independent or creative work in Geometry, a fact too little realised." Now this appears to me to be a very important paragraph. Is the free play of thought hindered by the mechanical difficulty of handling instruments? Personally, I believe it depends absolutely on the child. I daresay a clever child would be far better without all these aids. If she were obliged to use her imagination and exert herself from the beginning, we should turn out a few more good mathematicians, but we do not consider the clever girl nowadays, our teaching is for the average and the stupid. If a girl suggests that she can draw a $\triangle$ like the one on the board if she measures the 3 $\angle$s, a good part of the class will not be convinced until they have actually tried with their instruments, as the Board recognises in paragraph 35, where it states that results perceived by intuition should be fixed in the mind by construction. It is difficult to tell how the mind is working until one has the results of that working on paper. In Solid Geometry one realises the difficulty the girls have of imagining that which they cannot draw. I suppose the right course to adopt would be to get the girls to picture the figures to themselves. I must own that when I try this I feel that a few girls in the class follow ; but it is too much to expect of the rest, and they are wasting their time.

We are told in paragraphs 65 and 66, and again in 77, that the real stress of the work in deductive Geometry lies not on logic nor on the framework, but on the training of geometrical eyesight—for instance, in training the girls to pick out which $\triangle$s to compare in the figure as a kind of puzzle. The progress of Geometry depends much more on the special training of the eyesight or intuition than on logic ; the perception of the fact often or even generally precedes the consciousness of its reason, the logic often comes in after the intuition to substantiate and control it. If the teacher recognises the importance of this training of eyesight, it will clearly affect his blackboard work.

We see in paragraph 66 that he will build up the figure in front of the class, and make a fresh start each time the proposition is repeated. He will draw the figures differently each time. He will train his class to notice the modification of a figure, *e.g.* as a parallelogram becomes a rectangle, a square, or a rhombus. In discussing a parallelogram he will draw also a trapezium, and make his class see whether or when the proof breaks down. He will not waste time drawing his figures accurately with ruler and compass, nor allow his pupils so to do at this stage.

I gather then that we are to lay more stress on the training of the geometrical eyesight or intuition (par. 77) than on logic, that we are to use figures as a help, sometimes accurate, especially when new ideas are introduced, such as locus and envelope, but more often rough figures, our ideal being so to train them that they may be able to imagine figures and constructions, and that the free play of thought may not be hindered by the necessity of reliance on physical eyesight.

On the subject of marking written work (par. 74). The circular states that some teachers mark far too leniently, giving marks for details when the whole point of the proof is missing; that mechanical methods of marking, a line at a time, are fatal in this subject. I must say I was surprised to hear that it is necessary to state this. But the circular goes on to say, "Pupils should not even be allowed to begin writing out introductory matter, hypothesis and construction, till they have seen their way to the real point." What does that mean exactly? It seems to me that the first thing is to get the girls to be clear as to what is given and what has to be proved, and if the teacher thinks it helps the children to write that down, why should they not do so? If she waits till she is quite certain that every girl has seen her way to the real point—well and good, but how is she to be sure in a class of thirty until they have tried to work out the theorem?

We should all welcome any suggestions that lessen the laborious operation of the full correction of geometrical work. The Board suggests that "teachers should train their pupils to produce figures with such marks on them as will show at a glance whether they have seized the solution. A teacher who has trained his own eyes can then see quickly whether his class is doing sound work, and it is then only necessary to have solutions completely written out now and then." I should be glad to hear the opinion of members on this subject. From my own experience, I have found that at the beginning of the deductive Geometry one cannot be too particular to train the children to write out the theorems carefully, and it is fatal to suggest that they may satisfy their teachers with anything less than the whole theorem. With girls who are of matriculation age and who are revising I have allowed proof only, but I have found it fatal unless I repeatedly tell them that it is quite wrong to omit the rest, and the omission is allowed solely because I wish them to have more time for riders. One girl who came to us, and who had been allowed to write proof only, was never cured, even though she stayed with us for Intermediate Arts. To the end she wrote proof only. I should never let elder girls write part of the rider or theorem—either marked figure only or the whole thing.

In paragraph 75 stress is laid on the importance of training pupils in the accurate use of English both orally and in writing, and that no subject provides better opportunity than Geometry for training in stating exactly what is meant in clear and terse English. In the next paragraph we are told that a few simple precautions save a great deal of difficulty, confusion, and useless repetition. The use of letters in oral work is often a great hindrance, and should be kept down to a minimum, for they make it hard for the rest of the class, and even for the teacher himself, to follow what an individual pupil is saying. Descriptive terms (right and left, upper and lower, and the like) may often be used with advantage where actual marking or pointing is inconvenient. Where letters have to be used, angles should, whenever possible, be named by a single letter. The point is that the class should be enabled to follow as easily and quickly as possible, and artificial difficulties should not be put in the way merely because letters will some time or other have to be used in written work. Having been fighting against slovenliness in any form for all the years I have taught Geometry, this paragraph comes as somewhat of a shock. Again, with elder girls who have already had good training, I see no reason why, if there is only one angle at A, they should not say the angle A both in written and oral work, but with little ones this is risky. It is unwise to tell them they *must* do one thing one time and *may* do another at another. With them the thing is either

right or wrong. Of course, when a beginner has discovered in class how to do a rider and is most anxious to let us all know, I allow the left-hand and right-hand angle, etc., for it is very annoying to be stopped by details when one has made a grand discovery. I find the majority of the class does not follow what she is talking about; but when I get her to repeat it carefully with the proper letters, the class has time to follow. Also it is, of course, of the utmost importance that every girl in the class knows exactly what is given and what has to be proved. For this reason it is as well to write on the board exactly what is given and what has to be proved and to get the girls to repeat it, and they cannot do that very satisfactorily without letters. Also, I do not see that this is consistent with the remarks on the importance of training in English.

I feel that the circular is very suggestive and helpful; in fact, I must own that on first reading it there were many matters with which I did not agree, but the point of which I have seen after careful reading. Still there are many suggestions which one hesitates to adopt without discussion with other teachers. On the surface some of the points I have raised may appear to be somewhat unimportant and trivial details, but to a practical teacher of elementary Mathematics they are by no means unimportant, and the Board has recognised this by going into details in this circular.

M. J. PARKER.

THE DISSECTION OF RECTILINEAL FIGURES.

BY W. H. MACAULAY, M.A.

WHEN my previous paper on the "Dissection of Rectilineal Figures" was published (*Mathematical Gazette*, October, 1914, vol. vii. p. 381), I had not seen Mr. H. M. Taylor's paper in the *Messenger of Mathematics*, 1905, vol. xxxv. p. 81. He gives the four-part dissection of a pair of triangles, and two varieties of this; the three dissections being all of one type. He also gives the three-part and other dissections of a pair of parallelograms. He also gives a four-part dissection of a triangle and a parallelogram (equivalent, by projection, to that of a triangle and a square) which, for the case of an equilateral triangle and a square, was published in 1905 by Mr. H. E. Dudeney. This dissection, which I did not know before, is also applied by Mr. Taylor to the case of a quadrilateral and a triangle with a side equal to a diagonal of the

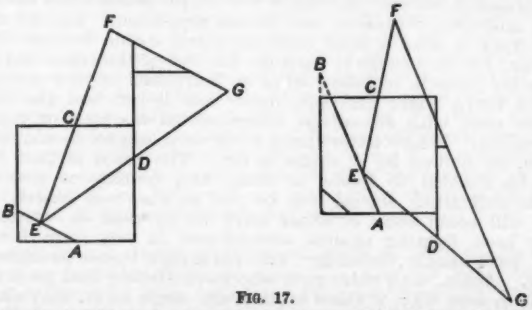

FIG. 17.

quadrilateral. As I have already given two types of dissection of a triangle and a square, with four parts in their fundamental case, which I have called the first and second types, I will call this, which is shown in Fig. 17, the third type.

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All three types apply equally to a square or to a parallelogram, but it is most convenient to discuss them for the case of a square. The third type of dissection gives four parts if the triangle can be placed within a square of four times its area, with one of its angular points at the middle point of a side of the square. With more parts and broken lines it is applicable to any triangle.

In Fig. 17 a six-part dissection is shown, as well as the fundamental case with four parts. The construction depends only on drawing, from a middle point of a side of the square, a line AB to an adjacent side (produced if necessary) equal to a half side of the triangle. This can always be done, often in several different ways. To complete the construction, take CD equal and parallel to AB, C being the middle point of another side, then if a quarter of the triangle is constructed with base CD, its vertex E lies in AB, produced if necessary; thus the whole triangle EFG can be drawn. Mr. H. W. Richmond has investigated, for triangles of all shapes, the number of four-part dissections of all three types which actually have four parts. He finds incidentally that the third type provides a four-part dissection, or more than one, for those triangles which have a four-part dissection of the first or the second type, and for all the triangles approaching to being equilateral for which the first and second types of dissection do not give four parts. The results for a triangle and a parallelogram can be deduced by orthogonal projection. The triangle given as an example in my former paper, p. 387, as having with a square two four-part dissections of the first type and one of the second, has also four of the third type. Also, as I shall now point out, the two dissections of the first type are not merely single dissections, but each really represents a continuous range of dissections. The first type of dissection of a triangle and a square has, in fact, the rather unusual peculiarity that the dissection can be varied

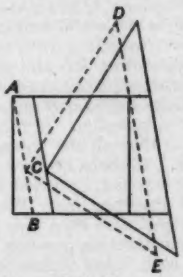

FIG. 18.

continuously in the way shown in Fig. 18. Here the dotted line shows a dissection in the form given in my former paper, p. 387, Fig. 12; but we obtain a continuous series of different dissections by moving the line AB and the triangle CDE into parallel positions, such as that shown by the full lines.

Since the publication of my previous paper, I have found that there is a four-part dissection of a pair of pentagons of equal area, each with two sides equal and parallel. This is important, because it generalises several other dissections, and coordinates them, besides leading at once to an eight-part dissection of a pair of quadrilaterals. The dissection is shown in Fig. 19, in which the pentagons are placed with corresponding lines parallel. Each pentagon has one side which I call its base, and two sides adjoining the base equal and parallel, and two other sides which may be said to be opposite the base, and meet at the vertex. I deal only with pentagons whose sides do not cross, that is to say, which enclose a single area; and it will be found that it is necessary to exclude pentagons which have a base angle re-entrant, and pairs of pentagons such that the area of each exceeds half the rectangle

contained by the two bases. Subject to these restrictions, a dissection of the type in question always exists, either with four parts which is the fundamental case, or with more parts and broken lines. Any of the angles, except those at the base, may be re-entrant. In Fig. 19 AB and CD are the bases, E and F the vertices, and G, H, J, K the middle points of the sides opposite the bases. The dissection depends on the possibility of drawing from H to AB, produced if necessary, a line HL equal to $\frac{1}{2}CD$, and from K to CD, produced if necessary, a line KM equal to $\frac{1}{2}AB$. This can be done if the area of a pentagon does not

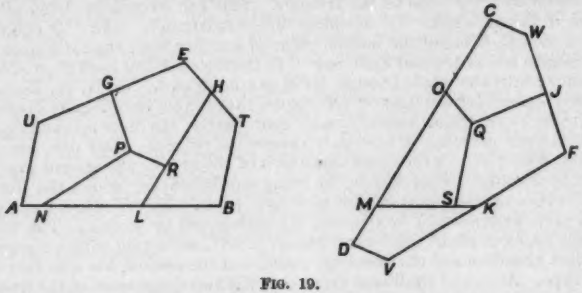

FIG. 19.

exceed half the rectangle contained by the bases. We then get identical parallelograms, $HLNG$ and $JKMO$, of area equal to half that of a pentagon. The point P is found by drawing NP, GP equal to KF, JF; and R is taken, so that HR is equal to OC. Similarly, Q and S are determined. It is clear that in Fig. 19 we get in this way a successful four-part dissection of the pair of pentagons. We get another dissection, with the same parallelogram, by drawing lines GN and JO instead of HL and KM; and we get two more dissections by using the parallelogram obtained by reversing $HLNG$, so as to get obtuse angles instead of acute angles at H and J. Thus there are altogether four dissections of the type in question. If there are no re-entrant angles, the condition for four parts is that all the points L, N, M, O lie within the bases, and not outside them. If there are re-entrant angles, we have still four parts if this condition is satisfied, provided that none of the sides of the pentagons happen to meet the dissecting lines. When the sides of the pentagons meet the dissecting lines, we get more than four parts; but, though complicated cases may occur, the construction presents no difficulty. We may still count the dissecting lines in each figure as four in number, provided that, as

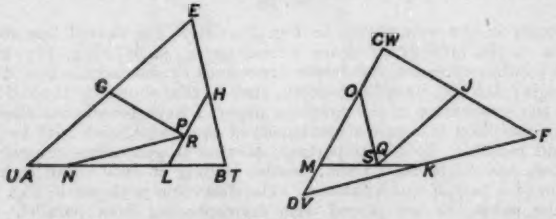

FIG. 20.

in other dissections, we allow a line to be made up of several parallel portions. The chief interest of this type of dissection lies in the fact that it coordinates other types, and from this point of view the fact that the dissection exists only for a certain range of pairs of pentagons is not important. The reason of

this is that the limitations depend chiefly on the fact that each pentagon has only a single base; accordingly, in applications to figures which do not possess any one side that can be distinguished from the other sides, the limitations are apt to disappear or lose their importance. Let us now consider some particular cases.

Suppose the lengths of the equal and parallel sides of both the pentagons in Fig. 19 to be zero; the dissection then becomes the four-part dissection of a pair of triangles, which was discussed in my former paper, and is shown in Fig. 20. The letters in this figure (and in subsequent figures) are the same as in Fig. 19, and show the points which coincide; they also show in each case the identical parallelograms *HLNG* and *JKMO*.

Next, let us suppose each of the vertical angles of the pentagons to be two right angles. We then get the four-part dissection of a pair of parallelograms which is shown in Fig. 21. This is not, like some four-part dissections of a

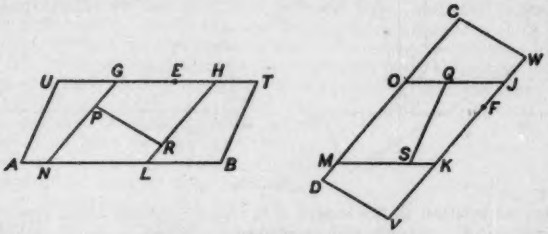

FIG. 21.

pair of parallelograms, a modification of the three-part dissection (see Fig. 10, p. 386), but differs from it in type. Next, suppose one pentagon to be a triangle, as in Fig. 20, and the other a parallelogram, as in Fig. 21; we then get what I have called the first type of four-part dissection of a triangle and parallelogram (or square) in its general form, as shown in Fig. 22. There are

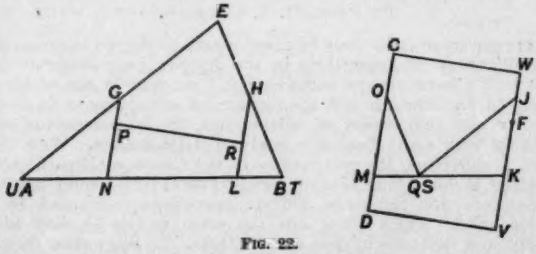

FIG. 22.

various other particular cases of the pentagon dissection. Thus the angle at *U* may be made equal to two right angles, so that one of the pentagons becomes a quadrilateral with two parallel sides. A pair of such quadrilaterals has a four-part dissection.

What I have called the third type of four-part dissection of a triangle and parallelogram (or square) is not a particular case of the pentagon dissection, but is derivable from it by one additional cut in the following way. Let the first pentagon be a quadrilateral with two sides parallel, the angle at *U* being taken equal to two right angles; and let the second be a triangle with the side *VF* equal to *AB*, so that the points *D*, *V*, *M* coincide. The pentagon dissection, which the coincidence of dimensions reduces to three parts, is

shown in Fig. 23, with the same lettering as before. Now make a cut through the middle point of AB and parallel to ET and the corresponding cut in the triangle; this cut converts the first pentagon into a parallelogram whose dimen-

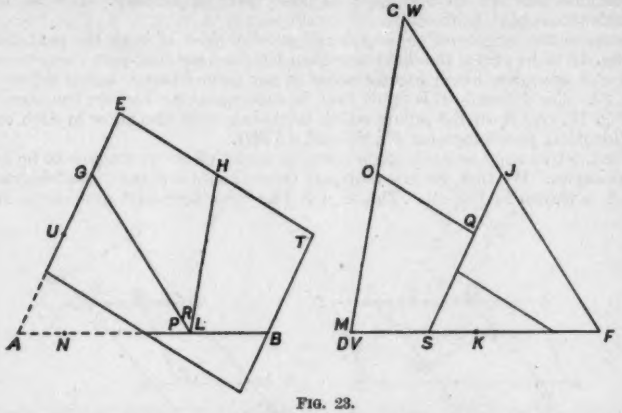

FIG. 23.

sions have no relation to the length AB, and we get the third type of four-part dissection of a triangle and parallelogram, which is shown in Fig. 23 in full lines.

(To be continued.)

THE DISCOVERY OF LOGARITHMS BY NAPIER.*

BY PROF. H. S. CARSLAW.

§ 1. A LITTLE more than three hundred years ago Napier announced to the world his discovery of Logarithms in the *Mirifici Logarithmorum Canonis Descriptio*.† "I have always endeavoured," we read in one of his works, "according to my strength and the measure of my ability to do away with the difficulty and tediousness of calculations, the irksomeness of which is wont to deter very many from the study of mathematics. With this aim before me, I undertook the publication of the Canon of Logarithms which I had worked at for a long time in former years; this canon rejected the natural numbers and the more difficult operations performed by them, substituting others which bring out the same results by easy additions, subtractions and divisions by two and by three."‡ Soon after the publication of the Canon, what he calls "another and better kind of logarithms" occurred to him,§ practically logarithms to the base 10, as we now know them, and a similar change in his system also suggested itself to Briggs, then Gresham Professor of Geometry in London, one of the first to recognise

* The publication of this paper has been unfortunately delayed. The author intended it to appear at the time of the Napier Ter-Centenary Celebrations.

† (Edinburgh, 1614.) English translation by Wright (London, 1616), and by Filipowski (Edinburgh, 1857).

‡ Quoted in Macdonald's translation of the *Mirifici Logarithmorum Canonis Constructio*, p. 88. (Edinburgh, 1889.)

§ Natural logarithms, Napierean logarithms as they are sometimes called, are not the logarithms known to Napier.

the immense value of Napier's discovery. In the interval between 1614 and Napier's death in 1617, they had devised several methods for the construction of the new Tables; but owing to Napier's failing health he had decided to leave the actual computation to Briggs. As early as 1617 Briggs was able to publish the logarithms of the numbers from 1 to 1000 to eight places. In 1624 he followed this with the numbers from 1 to 20,000 and 90,000 to 100,000 to fourteen places. In 1628 Vlacq, a Dutchman, published the logarithms of all numbers from 1 to 100,000 to ten places, along with a Table of the Logarithms of the Trigonometrical Ratios for every minute, to the same number of places. These Tables of Briggs and Vlacq have never been superseded—their work was done for all time. They contain errors which have gradually been discovered and corrected, but all the Tables that have been published since their time, with one single exception, have been copied, directly or indirectly, from their work.*

§ 2. Logarithms, as Napier first understood them, and even logarithms in the later form agreed upon by him and Briggs, did not appear to their inventor in the light in which we now regard them. The modern exponential notation a^n was not yet invented, and it was not for more than a hundred years that the idea of a logarithm as the index of the power of the base found a place in works on algebra. Indeed, in the original system of Napier, there is no mention of a base at all; and in the modified and improved system, though as a matter of fact it does in a sense consist of logarithms to the base 10, no stress is laid upon that point.

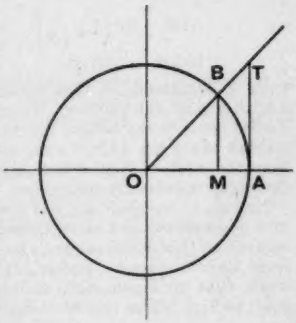

FIG. 1.

Before we can understand Napier's work, it is necessary to call to mind that his object was to render calculations with sines, cosines, etc., especially the calculations of the astronomer, an easier matter. To the mathematicians of his time, the trigonometrical functions were not *ratios*, but *lines*, or the measures of lines. The *sine of the arc AB* (Fig. 1), as they would put it, was the *line BM*. Its *cosine* was the *line OM*, etc. In the Trigonometrical Tables, which had been calculated by German mathematicians to an enormous degree of accuracy, these functions were given as integers, the radius being the sine of 90°, or the *sinus totus*. When additional accuracy was required, the radius was chosen proportionally large. In the Trigonometrical Tables used by Napier, it was 10^7; in those used by Briggs and Vlacq, it was 10^{10}.† Before Napier's time, it must be remembered, the notation for decimal fractions had not been invented. Indeed this is another invention of Napier's. At any rate, it appears in his work for the first time in its modern form.‡

§ 3. It is sometimes stated that Napier's Logarithms were obtained from the co-ordination of two definite series, an arithmetical and a geometrical. For instance, Cajori, in his recent paper on the "History of Logarithms,"§ says:

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* Cf. Glaisher, *Phil. Mag.* (4), vol. 44, p. 291 (1872).

† It is because Briggs and Vlacq used Tables of the Trigonometrical Ratios in which the radius was taken as 10^{10} that the characteristics 10, 9, 8, etc., appear in the Tables of Logarithmic Sines, Cosines, etc. In common with many other writers of text-books on Trigonometry, I regret to say that I have been guilty of the mistake of saying that 10 is added to the logarithms of the trigonometrical ratios to avoid the inconvenience of printing negative characteristics.

‡ Cf. Macdonald, *loc. cit.* p. 88.

§ *American Mathematical Monthly*, vol. xx. p. 7 (1913). Also Tropfke, *Geschichte der Elementar-Mathematik*, vol. ii. pp. 148-150 (Leipzig, 1903).

"Letting $v=10^7$, the geometric and arithmetic series of Napier may be exhibited in modern notation as follows:

$$\begin{array}{cccccc} v, & v\left(1-\frac{1}{v}\right), & v\left(1-\frac{1}{v}\right)^2, \ldots & v\left(1-\frac{1}{v}\right)^n, \ldots. \\ 0, & 1, & 2, \quad \ldots & n, \quad \ldots. \end{array}$$

The numbers in the upper series represent successive values of the *sines*; the numbers in the lower series stand for the corresponding logarithms. Thus $\log 10^7=0$, $\log(10^7-1)=1$, and generally, $\log(10^7(1-10^{-7})^n)=n$, where $n=0, 1, 2, \ldots$."

This statement is incorrect. In Napier's Tables the logarithm of (10^7-1) is not 1. It lies between 1 and 1·000,000,1, and he takes it as the mean between these two numbers, namely 1·000,000,05.

If Napier had used these two series in the way named above, his work would have more closely resembled that of Bürgi than actually is the case. In the History of Logarithms, as in that of so many other discoveries in mathematics and in every branch of science, we find that more minds than one at the same time were moving towards the same goal, quite independently of one another. To Bürgi, in Prague and Cassel, it had occurred to draw up Tables in which the series

$$\begin{array}{cccccc} 10^8, & 10^8\left(1+\frac{1}{10^4}\right), & 10^8\left(1+\frac{1}{10^4}\right)^2, \ldots & 10^8\left(1+\frac{1}{10^4}\right)^n, \ldots \\ 10\times 0, & 10\times 1, & 10\times 2, \quad \ldots & 10\times n, \quad \ldots \end{array}$$

were co-ordinated, so that multiplication and division might be reduced to addition and subtraction. Bürgi, however, though he had calculated his Tables some years before the publication of Napier's work in 1614, did not publish them till 1620,* and, even then, in an incomplete form. By that time the work of Napier and Briggs was available, and the value of Napier's discovery universally recognised.

Napier, as we shall see, did make some use of the idea of the co-ordination of a geometrical and an arithmetical series, but when he comes to define his logarithms, his definition is quite independent of the series given in the extract from Cajori's paper. Indeed, it is one of the most amazing features of his work, that his kinematical definition of the logarithm should have suggested itself to him before the invention of the Calculus.

§ 4. We shall now describe the method followed by Napier, as explained by himself in the *Mirifici Logarithmorum Canonis Constructio*. This post-

* Bürgi's work was entitled *Arithmetische und Geometrische Progress-tabulen, sambt gründlichen unterricht, wie solche nützlich in allerley Rechnungen zu gebrauchen und verstanden werden sol* (Prague, 1620). But the promised explanation of the Tables is not contained in the volume. A MSS. with this explanation was found at Dantzig about 1850. The numbers of the A.P. he calls the *red numbers* and prints them in *red*. The numbers of the G.P., he calls the *black numbers*, and prints them in that colour. The product of any two *black numbers* can be obtained by taking the sum of the two corresponding *red numbers* and then reading off from the Table the *black number* which corresponds to that sum, multiplying this *black number* by 10^8. For intermediate numbers interpolation would be required, but the series is supposed to have been calculated so far as to make the rule of proportional parts applicable. An extract from Bürgi's Table is given below:

	28 000	28 500	29 000	29 500	and so on to 31 500
0	**1323 11129**	**1329 74308**	**1336 40811**	**1342 10655**	
10	 24362	 87605	 54175	 24086	
20	 37593	**1330 00904**	 67541	 37518	
30	 50826	 14204	 80907	 56952	
40	 64061	 27506	 94267	 64387	
⋮					

The figures in heavy type are Bürgi's *red numbers*.

humous work was published in 1619, two years after Napier's death, but it had been written, at least in part, before the publication of the *Descriptio*. In addition to the body of the book, comprising the account of the theory and construction of his Tables, it contains Napier's "Appendix as to the making of another and better kind of Logarithms," some propositions for the solution of Spherical Triangles, and Notes on them and on the Appendix by Briggs.

Napier begins by forming a Geometrical Progression of 101 terms, the first term being 10^7 and the common ratio $\left(1-\frac{1}{10^7}\right)$.

We denote the terms of this series by

$$a_0, a_1, a_2, \ldots a_{100},$$

and the common ratio by r_1.

Thus
$$\left.\begin{aligned} a_0 &= 10^7, \\ a_1 &= 10^7\left(1-\frac{1}{10^7}\right) = a_0 r_1, \\ a_2 &= 10^7\left(1-\frac{1}{10^7}\right)^2 = a_0 r_1^2, \\ &\vdots \\ a_{100} &= 10^7\left(1-\frac{1}{10^7}\right)^{100} = a_0 r_1^{100}, \end{aligned}\right\} r_1 = \left(1-\frac{1}{10^7}\right).$$

These successive numbers were easily calculated when the notation for decimal fractions was introduced, and the approximation carried only as far as was necessary for his purpose. They are contained in his *First Table*, which reads as follows:

The First Table.

$(a_0 =)$	1 000 000·000 000 0
	1·000 000 0
$(a_1 =)$	9 999 999·000 000 0
	·999 999 9
$(a_2 =)$	9 999 998·000 000 1
	·999 999 8
$(a_3 =)$	9 999 997·000 000 3
	·999 999 7
$(a_4 =)$	9 999 996·000 000 6
to be continued up to	
$(a_{100} =)$	9 999 900·000 495 0

§ 5. The last term of this First Table is a_{100}, or $a_0 r^{100}$. If we write $\rho_2 = \frac{a_{100}}{a_0} = r_1^{100}$, a new Geometrical Progression of 51 terms can be formed with a_0 and a_{100} for its first two terms, and ρ_2 for its common ratio.

We denote this series by

$$\beta_0, \beta_1, \beta_2, \ldots \beta_{50}.$$

Then
$$\left.\begin{aligned} \beta_0 &= a_0, \\ \beta_1 &= \beta_0\rho_2 = a_0 r_1^{100} = a_{100}, \\ \beta_2 &= \beta_0\rho_2^2 = a_0 r_1^{200} = a_{200}, \\ &\vdots \\ \beta_{50} &= \beta_0\rho_2^{50} = a_0 r_1^{5000} = a_{5000}, \end{aligned}\right\} \begin{aligned} a_0 &= 10^7, \\ r_1 &= 1-\frac{1}{10^7}, \\ \rho_2 &= r_1^{100}. \end{aligned}$$

Napier does not calculate the terms of this series, but he replaces it by one easier to compute, the terms of which will differ only slightly from the β's.

Instead of $\beta_1 = 9\,999\,900{\cdot}000\,495\,0$ he takes the number 9 999 900, which differs from it only by a small fraction.

We denote this number by b_1, and write

$$b_0 = a_0, \quad r_2 = \frac{b_1}{b_0} = \frac{9\,999\,900}{10\,000\,000} = 1 - \frac{1}{10^5}.$$

The numbers in Napier's *Second Table* are the 51 terms of the Geometrical Progression beginning with b_0, b_1, and with r_2 for its common ratio.

We denote these by

$$b_0, b_1, b_2, \ldots b_{50}.$$

Thus

$$\left.\begin{aligned} b_0 &= a_0, \\ b_1 &= b_0 r_2, \\ b_2 &= b_0 r_2^2, \\ &\vdots \\ b_{50} &= b_0 r_2^{50}, \end{aligned}\right\} \quad \left.\begin{aligned} a_0 &= 10^7, \\ r_2 &= 1 - \frac{1}{10^5}. \end{aligned}\right.$$

This Second Table reads as follows:

The Second Table.

$(b_0 =)$	10 000 000·000 000
	100·000 000
$(b_1 =)$	9 999 900·000 000
	99·999 000
$(b_2 =)$	9 999 800·001 000
	99·998 000
$(b_3 =)$	9 999 700·003 000
	99·997 000
$(b_4 =)$	9 999 600·006 000

and so on up to

$(b_{50} =)$ 9 995 001·224 804 *

§ 6. Next another Geometrical Progression is formed. This consists of 21 terms. The first term is the same as a_0 or b_0. We shall denote it by c_0. The second term c_1 is 9 995 000, very nearly the same as the last term in the Second Table. The common ratio of this series is $\frac{9\,995\,000}{10\,000\,000} = 1 - \frac{1}{2000}$, and is denoted by r_3. The terms of the series are

$$c_0, c_1, c_2, \ldots c_{20}.$$

Thus

$$\left.\begin{aligned} c_0 &= b_0 = a_0, \\ c_1 &= c_0 r_3, \\ c_2 &= c_0 r_3^2, \\ &\vdots \\ c_{20} &= c_0 r_3^{20}, \end{aligned}\right\} \quad \left.\begin{aligned} a_0 &= 10^7, \\ r_3 &= 1 - \frac{1}{2000}. \end{aligned}\right.$$

* Napier has 9 995 001·222 927. This mistake is carried forward into the *Radical Table* of § 11, and some of the Logarithms of his Canon are affected by it.

These terms are easily calculated. They form the first column of Napier's *Third Table*, and read as follows:

$$
\begin{aligned}
(c_0 =)\quad & 10\,000\,000{\cdot}000\,00 \\
& \underline{\quad 5\,000{\cdot}000\,00} \\
(c_1 =)\quad & 9\,995\,000{\cdot}000\,00 \\
& \underline{\quad 4\,997{\cdot}500\,00} \\
(c_2 =)\quad & 9\,990\,002{\cdot}500\,00 \\
& \underline{\quad 4\,995{\cdot}001\,25} \\
(c_3 =)\quad & 9\,985\,007{\cdot}498\,75 \\
& \underline{\quad 4\,992{\cdot}503\,74} \\
(c_4 =)\quad & 9\,980\,014{\cdot}995\,01
\end{aligned}
$$

and so on up to

$$(c_{20} =)\quad 9\,900\,473{\cdot}578\,08$$

§ 7. Napier's Third Table, in which the c's form the first column, contains 69 columns. The terms at the top of each column are the terms of the Geometrical Progression, of which the first two terms are c_0 and 9 900 000, a number differing only slightly from c_{20}.

We denote these terms by

$$d_0, d_1, d_2, \ldots d_{68}.$$

Then $\dfrac{d_1}{d_0} = 1 - \dfrac{1}{10^2}$, and is denoted by r_1.

Thus we have

$$\left.\begin{aligned} d_0 &= a_0 = b_0 = a_0, \\ d_1 &= d_0 r_1, \\ d_2 &= d_0 r_1^2, \\ &\vdots \\ d_{68} &= d_0 r_1^{68}, \end{aligned}\right\} \quad \left.\begin{aligned} a_0 &= 10^7, \\ r_1 &= 1 - \frac{1}{10^2}. \end{aligned}\right.$$

These form the terms in the *first row* of the Third Table. The columns each contain 21 terms in Geometrical Progression, the common ratio being r_2, or $1 - \dfrac{1}{2000}$.

The Third Table.

First Column.	Second Column.	Third Column.	Thence 4th, 5th, etc., up to	69th Column.
10 000 000·0000	9 900 000·0000	9 801 000·0000		5 048 858·8900
9 995 000·0000	9 895 050·0000	9 796 099·5000		5 046 334·4605
9 990 002·5000	9 890 102·4750	9 791 201·4503		5 043 811·2932
9 985 007·4987	9 885 157·4237	9 786 305·8495		5 041 289·3879
etc., continuously to	etc., descending to	etc., descending to		finally to
9 900 473·5780	9 801 468·8423	9 703 454·1539		4 998 609·4034

The common ratio of the terms in all these 69 *columns* is $\left(1 - \dfrac{1}{2000}\right)$.

The common ratio of the terms in the *top row* is $\left(1 - \dfrac{1}{10^2}\right)$.

§ 8. Napier thus obtained in his Third Table a set of numbers lying between 10^7 and very nearly $\frac{1}{2}10^7$, and these numbers form a set dense enough to allow this Table to be used in dealing with the sines of angles from 90° to 30°. The numbers are not exactly in a Geometrical Progression, but each of the 69 columns of this Table is a Geometrical Progression of 21 terms.

If we had kept to the one Geometrical Progression from the very beginning, as in the case denoted by the letters

$$\beta_0, \beta_1, \beta_2, \ldots \beta_{20},$$

—these terms all occurring in the continuation of the Geometrical Progression

$$a_0, a_1, a_2, \text{ etc.},$$

—we would have had

$$\left.\begin{aligned} \gamma_0 &= \beta_0 = a_0, \\ \gamma_1 &= \gamma_0\rho_3 = a_0 r_1^{5000} = a_{5000}, \\ \gamma_2 &= \gamma_0\rho_3^2 = a_0 r_1^{10,000} = a_{10,000}, \\ &\vdots \\ \gamma_{20} &= \gamma_0\rho_3^{20} = a_0 r_1^{100,000} = a_{100,000}, \end{aligned}\right\} \quad \left.\begin{aligned} a_0 &= 10^7, \\ \gamma_1 &= \beta_{20} = a_{5000}, \\ \rho_3 &= \rho_3^{20} = r_1^{5000}. \end{aligned}\right.$$

These would replace the terms in the first column of the Third Table.

Also the terms at the top of the 69 columns would now have been

$$\delta_0, \delta_1, \delta_2, \ldots \delta_{68},$$

where

$$\left.\begin{aligned} \delta_0 &= \gamma_0 = \beta_0 = a_0, \\ \delta_1 &= \gamma_{20} = \delta_0\rho_4 = a_{100,000}, \\ \delta_2 &= \delta_0\rho_4^2 = a_{200,000}, \\ &\vdots \\ \delta_{68} &= \delta_0\rho_4^{68} = a_{6,800,000}, \end{aligned}\right\} \quad \rho_4 = \frac{\gamma_{20}}{\gamma_0} = r_1^{100,000}.$$

The last term in this 69th column would be $\delta_0\rho_4^{69}$ or $a_{6,900,000}$.

If Napier had defined his logarithms by the correspondence between the two series

$$\begin{array}{cccccccc} 10^7, & 10^7\left(1-\frac{1}{10^7}\right), & 10^7\left(1-\frac{1}{10^7}\right)^2, & \ldots & 10^7\left(1-\frac{1}{10^7}\right)^n, & \ldots, \\ 0, & 1, & 2, & \ldots & n, & \ldots, \end{array}$$

the numbers denoted by the Greek letters above would have had for their logarithms the suffices of the corresponding a's. Thus the logarithm of the last term in the 69th column of the modified table would have been 6,900,000. [Cf. The Radical Table of § 11.]

This would be *the number of the ratios* required to obtain that term from the first term in the series, namely 10^7, whose logarithm would be zero.

This, in fact, is Napier's derivation of his word logarithm—*the number of the ratios*—and he obtained the idea from such a correspondence between an arithmetical and a geometrical series. But when he comes to define the term *logarithm*, he takes a new point of view altogether, and, though his logarithms nearly agree with those defined above, they do not do so absolutely.

§ 9. Expressed in modern language and notation, Napier's argument is as follows:

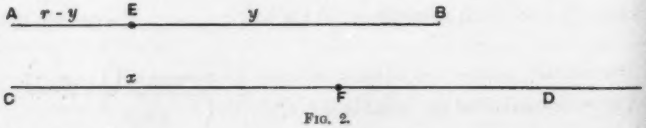


FIG. 2.

Imagine two lines AB and CD, AB of length equal to the radius, and CD unlimited in length. Let two particles start from A and C with the same velocity. But whilst the velocity of the particle which starts from C is to remain uniform, that of the particle which starts from A is to decrease in such a way that at any stage of its journey from A towards B, say at E, the velocity at E : the velocity at $A = EB : AB$.

When the one particle has reached E on AB, let the other particle be at F on CD.

Then the number which measures the line CF is called the logarithm of the number which measures the line EB.

With the notation of the Calculus (which it must be remembered was not discovered till about 1666), we can obtain the relation between the sine and its logarithm as follows :

Let $CF = x$ and $BE = y$ at the time t, and let the initial velocity of both particles be c times the radius r.

Then $$x = crt \quad \text{and} \quad \frac{d}{dt}(r - y) = cy.$$

From the second of these equations, $$\frac{dy}{dt} + cy = 0.$$

Then, using the initial conditions, $y = re^{-ct}$.

Thus $$x = r \log_e \frac{r}{y}.$$

If we write $\log_N y$ for Napier's logarithm of y, as defined above, we have

$$\log_N y = r \log_e \frac{r}{y}.$$

From the equation $\log_N y = r \log_e \frac{r}{y}$, it follows that if points P_1, P_2, ... are taken on AB such that BP_1, BP_2, etc., descend in G.P., the corresponding points on CD, namely Q_1, Q_2, ..., ascend in A.P. Further the logarithm of radius is zero.

Also $\log_N(uv)$ is *not* $\log_N u + \log_N v$, neither is $\log_N(u/v)$ the same as $\log_N u - \log_N v$.

But if $$z = \frac{uv}{r}, \quad \log_N z = \log_N u + \log_N v,$$

and if $$\frac{z}{r} = \frac{u}{v}, \quad \log_N z = \log_N u - \log_N v.$$

Finally, if $$\frac{u}{v} = \frac{u'}{v'}, \quad \text{then } \log_N u - \log_N v = \log_N u' - \log_N v'.$$

He demonstrated these theorems without the advantage of the Calculus methods, using the kinematical definition of a logarithm given in his words, as follows :

The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given sine.

§ 10. In obtaining the logarithms of sines he returns to the *Third Table*, which we have described above. He relies on the following theorems :

I. *The logarithm of any given sine is greater than the difference between radius and the given sine, and less than the difference between radius and the quantity which exceeds it in the ratio of radius to the given sine.*

II. *The difference of the logarithms of two sines lies between two limits ; the greater limit being to radius as the difference of the sines to the less sine, and the less limit being to radius as the difference of the sines to the greater sine.*

These theorems are now easy to establish with the aid of the logarithmic series. Napier obtained them direct from kinematical considerations.

Theorem I. Let s be a sine nearly equal to the radius r.

Then
$$\log_N s = r \log_e \frac{r}{s}$$
$$= r \log_e \left(1 + \frac{r-s}{s}\right)$$
$$= -r \log_e \left(1 - \frac{r-s}{r}\right).$$

Expanding $\log_e \left(1 + \frac{r-s}{s}\right)$ and $\log_e \left(1 - \frac{r-s}{r}\right)$, we have

$$(r-s)\frac{r}{s} > \log_N s > (r-s).$$

A close approximation to $\log_N s$ would thus be the Arithmetic Mean of $(r-s)\frac{r}{s}$ and $(r-s)$.

Theorem II. Again let s_1 and s_2 be two sines nearly equal to each other, s_1 being the greater.

We have
$$\log_N s_1 - \log_N s_2$$
$$= r \log_e \frac{r}{s_2} - r \log_e \frac{r}{s_1}$$
$$= r \log_e \frac{s_1}{s_2}$$
$$= r \log_e \left(1 + \frac{s_1 - s_2}{s_2}\right)$$
$$= -r \log_e \left(1 - \frac{s_1 - s_2}{s_1}\right).$$

Then it follows, as in Theorem I., that

$$r\left(\frac{s_1 - s_2}{s_2}\right) > \log_N s_2 - \log_N s_1 > r\left(\frac{s_1 - s_2}{s_1}\right).$$

(To be continued.)

MATHEMATICAL NOTES.

445. [X. 4.] *Two Problems on the Circle.*

These problems are original, though it is scarcely probable that they can be new. Their concrete nature and the fact that they lend themselves to graphical treatment makes it possible to discuss them with beginners, though the complete solution is less easy.

I. Being given two rods AOB, COD hinged together at O, and such that $AO.OB = CO.OD$ so that A, B, C, D are always concyclic, it is required to find the radius of the circle when the angle between the rods is θ and to determine the minimum circle for different values of θ.

[If $AO.OB = k^2$ and the mid-points of the rods are at distances l, m from O, it can be proved that

$$r^2 = k^2 + \frac{l^2 + m^2 - 2lm\cos\theta}{\sin^2\theta}.$$

The minimum value can be found from this, but is more easily obtained otherwise, the result once found being almost obvious.]

II. Being given a 45° set-square, ABC right-angled at B, an arc of a circle may be traced out by the point A by keeping AB, AC respectively in contact with two pins P, Q: it is required to prove that the length of the circular arc which may thus be described continuously

(i) increases uniformly as the distance PQ is increased, up to a certain maximum;

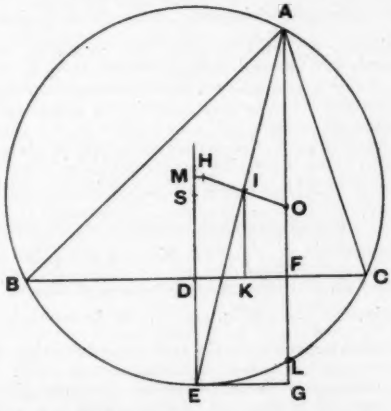
(ii) then suddenly drops to two-thirds of its maximum value;

(iii) then decreases proportionally to the decrease of $\frac{\lambda}{\sin\lambda}$ as λ decreases from $\frac{\pi}{2}$ to $\frac{\pi}{4}$.

It would be interesting to obtain other elementary concrete problems which, as this one does, afford a discontinuous graph. C. O. TUCKEY.

446. [K¹. 2. c.] *Feuerbach's Theorem.*

In the figure O is the orthocentre, I the in-centre, and S the circum-centre of ABC. EG is the tangent to the circum-circle at E, the middle point of arc BC. K is the point of contact of the in-circle with the side BC. OI is produced to H so that $IH=OI$, and HM is perpendicular upon SE.



$$\frac{DK.KF}{2Rr}=\frac{DK.KF}{EI.IA}=\frac{EG^2}{EA^2}=\frac{GL.GA}{2R.GA}=\frac{GL}{2R};$$

$$\therefore\ DK.KF=r.GL.$$

$$HM^2=(DK\sim KF)^2=EG^2-4DK.KF=EG^2-4r.GL.$$

$$SM^2=(SD+OF\sim 2r)^2=(SD+FL\sim 2r)^2=(\tfrac{1}{2}LA\sim 2r)^2$$
$$=R^2-EG^2-2r.LA+4r^2.$$

$$\therefore\ SH^2=R^2-4r(GL+\tfrac{1}{2}LA)+4r^2$$
$$=R^2-4rR+4r^2$$
$$=(R-2r)^2.$$

$\therefore$ circum-circle touches circle centre H and radius $2r$;

$\therefore$ N.P. circle touches in-circle. W. J. DOBBS.

447. [X. 8.] *Note on the Construction of String Models.*

It is sometimes desired to construct a string model of the hyperboloid of one sheet, either to illustrate the properties of ruled surfaces in general, or in connection with the isochromatic surfaces of a uniaxal crystal in polarised light. The following hints may be found useful for this purpose. If the surface is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, any point on its principal xy section is $(a\cos\theta,\ b\sin\theta,\ 0)$, where θ is the eccentric angle in the section. The generators through it are $\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\pm\frac{z}{c}$. Determine any other xy section by the auxiliary angle ϕ, such that the plane of the section is $z=c\tan\phi$. The generators cut this plane where

$$x=a\sec\phi\cos(\theta\mp\phi),\quad y=b\sec\phi\sin(\theta\mp\phi),$$

i.e. at points whose eccentric angles are $\theta\pm\phi$ in the section

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{z^2}{c^2}=\sec^2\phi.$$

Therefore the eccentric angle of an intersection changes at the same rate as the auxiliary angle ϕ of the section which is intersected. A further aid to neatness is to select equi-different values of θ, and preferably the various multiples of some exact submultiple of $\frac{\pi}{2}$. *E.g.*, if six strings to a quadrant are thought sufficient, θ would be given the values 0, 15°, 30°, etc., up to 345°. The boards for the end-sections should then be fixed where $\pm\phi$ is some multiple of the same angle, *e.g.* in the present case ϕ might be $\pm30°$. The result of such a choice is that each hook in the end-board serves for two strings, one for each set of generators, and this makes a neat arrangement. Moreover, the intersections of the strings lie in planes perpendicular to the z-axis, and this also gives a pleasing regularity; *e.g.* in the case suggested the strings would cross where $\phi=0$, $\pm7\frac{1}{2}°$, $\pm15°$ and $\pm22\frac{1}{2}°$, as well as at the end-boards.

Of course there is a corresponding theorem for the construction of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{k}$. In this case any point on the principal xy section is $(na,\ \pm nb,\ 0)$. The generator through the point, not passing through the origin, is $\frac{x}{a}-n=\mp\frac{y}{b}+n=\frac{z}{2nk}$. It is convenient to have the end-boards perpendicular to the x-axis, and the intersection of the generator with the plane $x=ma$ has $y=\pm(2n-m)b$, say $y=\pm rb$. Therefore the multiplier r in y changes at the same rate as the multiplier m in x. It is convenient to choose the generators to correspond to equidifferent values of n, *e.g.* all integral values. If, further, the end-boards be placed where m has an integral value, each hook will serve for two strings, one belonging to each set of generators. The intersections of the strings all lie in the planes $x=ma$, where m has all integral values. The intersections of the strings with any one such plane define points upon the parabolic section whose y's are all the alternate integral multiples of b.

A. O. Allen.

448. [L¹. 1. c.]. *Further Note on Pascal's Theorem.*

Prof. Genese has pointed out to me that the property of a quadrilateral quoted on p. 46, vol. vii., from the *Educational Times* (9681) is itself a simple case of Pascal's Theorem, the Pascal Line being at infinity. Since receiving his communication I have attempted to trace the occurrence of the theorem in works on Statics. Bow in his *Economics of Construction* quotes from Rankine, and I find in the second edition of *Civil Engineering* the now well-known theorem stated substantially as follows:—Let forces P, Q, R applied at the points A, B, C of a triangular frame-work ABC be in equilibrium,

their lines of action meeting in D, outside ABC, and let a triangle HKL be drawn with its sides KL, LH, HK parallel to the lines of action of P, Q, R. Then, if KO, HO be drawn parallel to AC, BC, LO will be parallel to AB. Now let HK be taken the same length as CD, and let the figure $HKLO$ be applied to $ABCD$ on the opposite side of CD, so that these two equal lines coincide, and we get *E. T.* 9681. This theorem has already been deduced statically in this manner, and shown to be a special case of Pascal's Theorem by Mr. P. J. Heawood in Note 75, when it is obtained from the consideration of a quadrilateral in equilibrium under forces along its sides, and a reference is made in Note 81 (*Elementary Geometrical Statics* by Mr. W. J. Dobbs, pp. 185-193). Prof. Crofton in his *Elements of Applied Mechanics*, after quoting Rankine's theorem and giving a statical proof, points out that it may be stated in the following *purely* geometrical form, and proved by pure geometry. If any three straight lines through the angles of a triangle ABC meet in a point D, and any other triangle HKL have its sides KL, LH, HK parallel to those three lines AD, BD, CD, then, if through H, K, L lines be drawn parallel to BC, CA, AB, they will meet in a point. E. M. Langley.

449. [I. 24. a.] *Note on approximate value of* π.

Let AOB be the diameter of a given circle, centre O. Produce OB to E so that $OB=BE$. Then, if P is a point on the circumference of the circle fairly near to A, and EP produced meets the tangent at A in Q, the arc $AP=AQ$. This follows from the formula of Snellius, $\theta=3\sin\theta/(2\theta)$.

Taking $AOP=\pi/6$, and assuming that $\sin\pi/6=1/2$ and $\cos\pi/6=\sqrt{3}/2$ ($\sqrt{3}=19/11$), we have $\pi=6\times3/2\times22/63=22/7$. R. F. Davis.

450. [X. 10.] *A Mathematical Recreation.*

I saw recently a card trick which, on analysis, rested on the proposition enunciated below. I do not know if it has been previously published, but it was new to me. Here is the proposition:

Remove from an ordinary pack of cards the court cards. Arrange the remaining 40 cards, faces upwards, in suits, in four lines thus: In the first line, the 1, 2, ... 10 of suit A; in the second line, the 10, 1, 2, ... 9 of suit B; in the third line, the 9, 10, 1, ... 8 of suit C; in the last line, the 8, 9, 10, 1, ... 7 of suit D. Next take up the first card of line 1, put below it the first card of line 2, put below that the first card of line 3, and put below that the first card of line 4. Turn this pile of four cards face downwards. Take up the four cards in the second column in the same way, turn them face downwards, and put them below the first pile of four cards. Continue this process until all the cards are taken up. Ask someone to mention any card. Suppose the number of pips on it is n. Then if it is of suit A, it will be the $4n$th card in the pack. If it is of suit B, it will be the $(4n+3)$th card in the pack. If it is of suit C, it will be the $(4n+6)$th card in the pack. If it is of suit D, it will be the $(4n+9)$th card in the pack. Hence by counting the cards in the pack, cyclically if necessary, the card desired can be turned up at the proper number. Any one with a liking for such recreations can alter the form of presentation in various ways, and a full pack can be used if desired.

W. W. Rouse Ball.

REVIEWS.

The Theory of Proportion. By M. J. M. Hill. Pp. xx+108. 8s. 6d. net. 1915. (Constable & Co.)

This treatise will be read with great interest and profit, because the author has not only taught the subject for a number of years, but has devoted a great deal of time and labour to the task of putting the theory in a form at once strict and intelligible. In this he has attained a marked success, and

it is unlikely that any great improvement on it can be made, at least in relation to the ordinary pupil.

Briefly, the procedure is as follows. First, we have a discussion (mainly after Stolz) of a set of magnitudes of the same kind, their multiples, and the ratio of two such quantities, when they are commensurable. This ratio is *defined* as a rational number; namely, if $nA=mB$, we have by definition $A:B=m/n$. Next a proof is given of the fact that there are geometrical quantities which are not commensurable; and this is followed by a very clear account of Dedekind's theory of irrational numbers. Then comes the definition of equal ratios, including a proof of Stolz's theorem; then three chapters giving the main propositions of the theory; then a chapter called "Applications of Stolz's theorem," which is practically an account of the method of exhaustion, with applications to circles, tetrahedra, etc.; and finally, a commentary on Euclid's fifth book.

There are a great many interesting points on which one is inclined to make remarks; only a few of them can be considered here. First of all, there is the psychological question as to the process by which mankind, and particularly the Greeks, arrived at a complete theory of ratio and proportion. In the absence of sufficient evidence, we can only arrive at vague and imperfect conclusions, but some inferences seem fairly certain. After the notion of equality, that of one quantity being double or half of a like quantity was acquired; and apparently this was, for a long time, the only consciously realised application of ratio. Of course, in commerce and the arts, multiples of quantities were constantly used, but they were determined by counting, and such evidence as there is seems to show that acquaintance with rational fractions as ratios, or more accurately as "parts" (*μέρη*), even in the case of aliquot parts, was slowly acquired. This is illustrated by such awkward terms as *sesquiunx* and *sesquipedalis* in Latin (*sesqui*, alone, is only once found) or *ἐπίτριτος* in Greek. Sometime or other it must have been realised that if the price of a cubit of linen from a particular roll was (say) a drachm, then there was a certain relation between 3 cubits and 4 cubits of linen similar to that of the 3 drachms and 4 drachms which would have to be paid for them. In this way, a sort of notion of the proportion of two couples of like commensurable quantities would arise; and it is not improbable that this preceded the geometrical cases where all the four quantities are of the same kind, even when practical use was made of the properties of similar figures. Eventually, it would be seen that there are innumerable cases where we have simultaneously $nA=mB$ and $nA'=mB'$. Whether A' is of the same kind as A or not, we can express this relation by saying that A, B, A', B' (in that order) "form an analogy" or "are in proportion"; and on this definition we can construct a theory of proportion without any discussion of ratio, so long as we confine ourselves to commensurable quantities. How did the term "ratio" arise? Probably by noting cases where $nA=mB$, but $nA'<mB$. Here it would be inferred that $nA>nA'$, and hence that $A>A'$. This being so, it might be said that since $A>A'$ absolutely, $A>A'$ *in relation to* B; and by a change of expression, we might say "the ratio of A to B is greater than the ratio of A' to B." This practically agrees with Prof. Hill's conclusion, p. 17. In a similar way, from $nA>mB$ but $nA'<mB$, we have $A>A'$, and the same conclusion about ratios as before. To make a strict theory out of this, we must show that we cannot have simultaneously such cases as $mA>nB$, $mA'<nB'$, $pA<qB$, $pA'>qB'$. It is just possible that Euclid's *arithmetical* theory of proportion arose from this; for the inequality $mq>np$, which follows from $mA>nB$, $pA<qB$, leads, if ϵ is any unit, to $m\epsilon/n>p\epsilon/q$, and this can be brought into connexion with Euclid's seventh book. However that may be, as soon as it was discovered that the length of the diagonal of a square is incommensurable with its side—and this was done when the Greeks were fully conscious that a square with one diagonal drawn is in some comprehensive sense "similar" to any other square with one diagonal drawn, and when it could be verified experimentally that a right-angled triangle with sides 3ϵ, 4ϵ, 5ϵ is similar to one with sides $3\epsilon'$, $4\epsilon'$ $5\epsilon'$ whatever the units ϵ, ϵ' may be, and that in this case the ratios of corresponding sides are equal—it would be almost inevitable to suppose that the diagonal and side of any one square and the diagonal and side of

any other square are in proportion, although the terms are incommensurable. The question now was to frame a definition of proportion which would apply both to the commensurable and to the incommensurable case; and this led, in a way which seems impossible to trace, to the invention of the relative scales of equimultiples, of which De Morgan's case of the trees and the railings is a classical illustration.

Here I venture to say something about Stolz's theorem. I must say that its importance, or at least its novelty, seems to me to be exaggerated. Euclid was quite aware that if A, B are incommensurable, the case $mB=nA$ cannot arise, so that the relative scale of (A, B) cannot give any coincidences; hence, in this case, for a proportion $A:B::A':B'$ we must establish the correspondences $mA \gtrless nB$, $mA' \gtrless nB'$ for all values of m, n and such relations as $mA=nB$, $mA'=nB'$ will not occur. But he meant to frame his definition of equality of ratios in such a way as to cover all cases, and therefore $mA=nB$ is included as a possible case, though it need not, and, in general, will not occur.

In this connexion I wish to say that I entirely disagree with Max Simon's assertion (p. 93) that "they (the Greeks) possessed a notion of number in all its generality." On the contrary, whatever the Greeks of Euclid's time meant by the ratio $A:B$, when $nA=mB$, I am very doubtful whether they ever thought of it as equivalent to the number m/n. And I am convinced that they never thought of the ratio of the diagonal of a square to its sides as the number $\sqrt{2}$; otherwise a great deal of Euclid's Elements would be superfluous; and superfluity is a defect to (or from) which the Greek mind is constitutionally averse. It is true, of course, that, on the basis of what the Greeks did know about ratios, it is possible to construct (with the Cantor-Dedekind axiom) a strict theory of number applied to geometrical quantities; but this is quite a different thing. A man may have all the materials for building a cathedral: it does not follow that he will build it; still less that his building will be perfect. Prof. Hill's reference to that most original mathematician, Michael Stifel (p. 93), should be noted.

As to Euclid's definitions, Book V. The third is absolutely silly, as it stands, for the only meaning I can make out of it is "the ratio of two homogeneous magnitudes is their relation with respect to quantuplicity, whatever that relation may be." Definition 4 restricts definition 3 (such as it is) by saying that the relations of quantuplicity must be such that the relations $mA>B$ *and* $nB>A$ must be satisfied by some integral values of m, n. (One, of course, may be taken to be 1, on Euclid's assumptions; *i.e.* either $A>B$, or $B>A$.)

What Euclid does not see is that we can define the symbols $>$, $=$, $<$, with regard to ratios without defining the term "ratio" itself; just as we can define the equality of two segments AB, CD without defining the term segment. In our definition we are treating "multiple" and "equimultiple" as more primitive terms than "ratio."

A former pupil of mine, not specially brilliant, but named Newton, pointed out to me that the theory of proportion might be based, not on relative scales of equimultiples, but on relative scales of submultiples; that is to say, $A:B=C:D$ if for all integers m, n

$$\frac{1}{m}A \lessgtr \frac{1}{n}B,$$

according as

$$\frac{1}{m}C \lessgtr \frac{1}{n}D.$$

From the Greek point of view, this theory is hardly so satisfactory as the classical one: because, although everybody admitted that any quantity could be multiplied to any extent, it was a question whether every quantity could be divided indefinitely into a number of equal parts.

In conclusion it may be remarked that the successive primes $p_1, p_2, p_3, \dots$, arranged in order of magnitude, form a system of similar magnitudes, but do not satisfy the laws of a set of "like" magnitudes, because we cannot satisfy

$$p_i+p_j=p_k$$

for all values of i, j.

G. B. MATHEWS.

Calculus Made Easy. By F. R. S. 2nd Edition, enlarged. (Macmillan & Co.)

This book is apparently written for science- and engineer-students who find it necessary to be able to understand such applications of the Calculus as they would meet with in their studies, but have not sufficient time to read a more systematic treatise on the subject.

Considerable ground is covered by the book, viz. the differentiation of all the standard functions, with applications to maxima and minima, and to speed and acceleration, and to various practical problems in physics and engineering, followed by a fairly full but rather more hasty treatment of integration, finishing with a short excursus into the solution of a few important types of differential equations. Just before the answers at the end there is a useful table of standard forms of integrals and differential coefficients for reference.

The explanations are of the colloquial type, mixed with jokes on the folly of mathematicians and others, which it is to be presumed the author has found useful in stimulating the ardour and courage of his students. Professor Perry in his books makes use of the same odd stimulus. It is a curious psychological fact if such banter does overcome intellectual diffidence and encourage effort.

It is a pity that the explanations are in some cases more colloquial than accurate: for instance, at the very beginning, d is said to mean "a little bit of," whereas in the subsequent pages it is quietly treated as "a little bit at the end of," or "a little addition to," which may be either positive or negative according to circumstances. The temptation to call it "a little bit of" occurs in integration when considered as a summation, but the temptation can be resisted, for, to borrow and modify the author's illustration, $\int dx$ means the sum of all the bits by which x *grows* (positively or negatively) from some initial value to such stature as at any point it has attained. It is quite true that it is also, as the author says, "the sum of all the little bits of x," but only because each 'dx' in turn has been added to the previous collocation of orderly 'bits.' When we turn to p. 6 we find 'dx' used in its proper sense, and so, of course, in all the work on differentiation.

Other small blemishes are:

On p. 12. The top of the ladder being y feet up when its foot is x feet from the wall, the author says it will be $y-dy$ feet up when the foot is $x+dx$ feet from the wall, whereas it is necessarily $y+dy$ feet up, and the peculiarity is that dy is negative.

,, p. 6, $(dx)^2$ is described as being a bit of a bit of x^2; but

,, p. 19, $(dx)^2$ is said to mean a little bit of a little bit of x.

,, p. 31, Ex. (4). In the second part it is not clear whether h is to increase with r in accordance with the condition $r=h$, or whether h is to be constant, *i.e.* whether the function to be differentiated is πr^3 or $\pi r^2 h$. Moreover, in either case, when r is only a few inches, it is rather a stretch to take $dr=1$ inch as a *differential*; it gives a result somewhat in excess of the true value.

,, p. 104, line 7. It is impossible if a, b have the same sign, but not otherwise.

,, p. 109, lines 10, 11. The quantities should be written $\frac{dS}{dt}$, $\frac{dr}{dt}$, $\frac{dV}{dt}$.

,, p. 166. No emphasis is laid on the fact that θ must be in radians, nor any explanation as to why we may take $\sin \frac{1}{2}d\theta$ as being the same as $\frac{1}{2}d\theta$.

,, p. 195, foot-note to top line. This is very confused and confusing, and needlessly so. x^2dx is $d(\frac{1}{3}x^3)$, or, more generally, $d(\frac{1}{3}x^3+c)$;

$$\therefore \int x^2dx = \int d(\tfrac{1}{3}x^3+c), \text{ which } = \tfrac{1}{3}x^3+c \text{ in exactly the same way as } \int dy = y.$$

In spite of such blemishes, the work is very sound as a whole, and many of the examples are of considerable interest, and it is not surprising to see

that the book has met with a good deal of success, as it carries the practical student to a very useful point.

There is a slip at foot of p. 170. We should read $\frac{d^2(\sin\theta)}{d\theta^2} = -\sin\theta$.

A. LODGE.

Geometry of Four Dimensions. By H. P. MANNING. Pp. ix + 348. 8s. 6d. net. 1914. (New York: The Macmillan Company.)

Although the study of Geometry of four and more dimensions was really inaugurated by British mathematicians—Cayley, Sylvester and Clifford—it is now almost neglected in this country, and is treated only in a subsidiary fashion as a mode of representation of analytical relations, and not as a field of geometrical enquiry. Thus, the latest edition of the *Encyclopedia Britannica*, which gives a detailed account of the non-euclidean geometries, dismisses the "fourth dimension" in four lines. For this reason—or is it the other way round?—the subject has never been presented systematically to the English reader. We therefore owe a debt of gratitude to Professor Manning for the preparation of this logical and comprehensive treatise, and for the care and thought which he has devoted to removing the difficulties which are inherent in the subject.

There are two classes of people whose interests have led them to a study of hyperspace: the mathematician and the philosophically (or spiritualistically) inclined layman. To the former the subject suggests mostly equations in n variables and a straining of geometrical language; the latter loves to imagine objects disappearing from closed rooms and knots tied on endless strings. In the book before us both of these aspects are ignored, and the subject is treated in the spirit of pure synthetic geometry. As the author explains in his introduction, there is much to be said for such a treatment; it certainly exercises the reader's powers of visualisation and deductive reasoning in a way which no manipulation of symbols will.

The scope of the book is limited to the elementary figures and the "round bodies," and deals with their descriptive and metrical relations. A characteristic feature is that the treatment is carried as far as possible without any assumption relating to parallel lines, so that the first five chapters, or 200 pages, apply equally well to non-euclidean space. To secure this result, however, many theorems (*e.g.* § 27, Th. 3, and § 53) have been expressed in a form which the reader, without a knowledge of non-euclidean geometry, will have difficulty in understanding. From the logical point of view there is an advantage in point of generality in dispensing with auxiliary aid which is not essential, but such a course tends to perplex the beginner. The exposition would be much easier to follow if at each stage the attention were confined at first to the euclidean case in which two coplanar lines either intersect or are parallel. But, after all, it the student wishes fully to understand the subject, especially the geometry of planes through a point, he must make up his mind first to acquire a knowledge of non-euclidean geometry.

The following is a brief summary of the contents. First comes a short introduction of twenty pages, which contains some valuable historical references. Chap. I. deals with the fundamental relations of lines, planes and hyperplanes, convex polygons, pyramids, hyperpyramids, hypercones and "plano-conical hypersurfaces" (or hypercones—"hyperconical hypersurfaces"—of the second species). The limits of the book have precluded any mention of hyperquadrics in general, or indeed of hypercones of either species with directing surface or curve other than a sphere or a circle. The interesting properties of ruled hypersurfaces are therefore excluded. Chap. II. deals with perpendicularity, and Chap. III. with the angles between two planes with only one point in common. This part of the subject is apt to present special difficulty to the reader who is accustomed to picture two planes as always cutting in a line and being fixed relatively to one another by a single dihedral angle. The configuration of two planes with only one point in common is analogous to that of two skew lines in elliptic space of three dimensions; these have two common perpendicular lines on which are measured the shortest distances between the two lines. The author establishes the existence of a common perpendicular to two skew lines by obtaining a sequence of points

and using Dedekind's axiom. The same procedure is repeated, *mutatis mutandis*, for two planes. The author's design is to exhibit for the analogous theorems in the more familiar region of three dimensions what exactly has to be proved and what assumptions are employed, so that the reader may have acquired confidence when he comes to apply the same methods in space of four dimensions. This should be remembered when the treatment is inclined to appear tedious. Then follows the theory of "isocline planes," analogous to equidistant or paratactic lines in elliptic geometry. This is taken up and completed in Chap. IV. after a discussion of symmetry, order and motion. Chap. V. treats further of the metrical properties of hyperpyramids and cones and the geometry of the hypersphere. Chap. VI. for the first time introduces parallels and elements at infinity, and describes the properties of prisms and cylinders. It is remarkable that the hypercube or 8-cell, which is perhaps the most familiar of four-dimensional figures, is mentioned for the first time on page 239. Chap. VII. treats of volumes and Chap. VIII. (the last) describes the six regular polyhedroids. There is a copious index and a very useful list of technical terms in English, French and German. The latter might have been usefully extended to include the Italian technical terms as found in the original edition of Veronese, and in the memoirs of Segre and others of the Italian school who have done so much to develop the projective geometry of hyperspace.

D. M. Y. SOMMERVILLE.

Motion of Liquids. By Lieut.-Col. R. DE VILLAMIL, R.E. Pp. viii + 118. 7s. 6d. net. 1914. (E. & F. N. Spon.)

Colonel de Villamil is always stimulating—and never more so than when most heretical. His latest work on the motion of liquids deals in a racy, but thoroughly workmanlike manner with the flow of liquids in the neighbourhood of plates, the phenomena of jets, the movement of liquids through apertures, the flow of water in rivers and canals, "negative resistance" in liquids, and curves of resistance. A special and valuable feature of his volume is a detailed discussion, with a wealth of numerical illustration, of the very much under-appreciated work of Dubuat and Duchemin. A large number of instructive drawings of lines of flow is given, together with several reproductions of Dr. Hele-Shaw's remarkable photographs of the flow of glycerine in two dimensions past various obstacles.

The book can with advantage be placed in the hands of an intelligent and inquiring student, and will certainly prove stimulating to all those who read—as Colonel de Villamil would have them read—in the spirit of Locke: "Reading furnishes the mind only with the materials of knowledge; it is thinking makes what we read ours."

The Dynamics of Surfaces. By L. MICHAELIS (translated by W. H. PERKINS, M.Sc.). Pp. xiv + 210. 4s. net. 1914. (E. & F. N. Spon, Ltd.)

The abnormally early age at which Tristram Shandy's misfortunes began has become proverbial, and the book before us is in like plight. Its title would lead an unwary reader to look for a discussion of the effects of air-resistance on aeroplane wings. It is, however, "intended to summarise the essential points in the theoretical treatment of surface phenomena, always keeping in view the object of applying the knowledge obtained for biological purposes," and, while containing accounts of very diverse surfaces and surface-phenomena, it contains very little dynamics, and that little by no means of the best quality. To criticise in detail the various physical errors and heresies which disfigure the volume would trespass too long on the reader's time and patience, and one or two examples must suffice. Surface-tension, for example, is uniformly described as a force, and it cannot be too strongly emphasised that surface-tension is a *force-per-unit-length*, and is no more a force than a foot is a velocity.

Again, in the definition of surface-tension we are told that it is measured by the "pull which is exerted . . . *by* any line in the surface of 1 cm. length. . . . Any irregular line exerts a pull which is proportional to its length." The exertion of force by a geometrical abstraction is distinctly a dynamical novelty!

The account given of the experimental methods for the measurement of surface-tensions is unfortunate in many ways. The "capillary-rise" and the "falling-drop" methods are those selected for detailed treatment, and the treatment of the latter method contains all those errors which are usually enshrined in the text-books on physical chemistry. We are told that the drop falls when its weight is sufficient to overcome the tension in its surface—which is either a piece of very slip-shod writing, or an elementary mistake in the theory of dimensions, involving, as it does, the equating of a force to a force-per-unit-length. The equation which gives the relation between the weight (W) of the falling drop, the surface-tension (T) of the liquid, and the radius (r) of the tube employed is given as

$$W = 2\pi r T.$$

Any experimenter misguided enough to employ this formula would find that his results for T would be about half their true value. As a matter of fact, the equation which best represents the results of experiment is

$$W = 3{\cdot}8 r T,$$

and by no stretch of imagination can 2π be regarded even as a first approximation to 3·8. The dynamical error involved in the deduction of the erroneous equation was pointed out over thirty years ago by Mr. A. M. Worthington, but generation after generation of text-book makers have industriously transcribed Quincke's original formula.

In discussing the surface-tensions of mixtures we are told that "it has not yet been found possible to represent the surface-tension as a function of the concentration." Nevertheless, immediately adjoining this statement is a table showing the variation of surface-tension with concentration for various organic liquids mixed with water, and anyone who cares to spend a few minutes plotting out the figures there given, will find that they lie on smooth curves which are satisfied by easily-found empirical equations. As a matter of fact, Duclaux, the authority cited for the figures, has suggested such equations to fit them.

Having got through the jungle of his dynamical introduction (not without a few cuts and bruises by the way), the writer is on surer ground. Clear accounts are given of various adsorption phenomena, and different empirical formulae for adsorption equilibrium are deduced and discussed. Passing on to the consideration of the surface as the seat of electrical forces, the author gives a resumé of the theory of metallic and electrolytic electrodes as worked out by Nernst, discusses the mechanical adsorption of electrolytes, the chemical effects of mechanical adsorption, the relation between the surface-tension and potential of capillary electrodes, and the relation between surface-tension and curvature—a point which has an important bearing on the theory of colloidal solutions. The volume concludes with a brief discussion of the kinetics of surface reactions.

We have emphasised the various mistakes made in dealing with the fundamental principles, since the volume definitely claims to expound principles rather than to collect large masses of isolated facts. Such an attempt should not be made without due training in, and a lively appreciation of the leading facts of dynamics; there is otherwise grave danger that the symbols of the infinitesimal calculus may be used in such a way as to give an idea of a precision that does not exist in reality—and muddled thinking in elementary dynamics is not likely to be clarified (though some devotees of certain branches of chemical science seem to think otherwise) by the use of differential equations.

ALLAN FERGUSON.

(i) **Plane Trigonometry.** Pp. 188. (ii) **Trigonometric and Logarithmic Tables.** By GEORGE WENTWORTH and DAVID EUGENE SMITH. Pp. 104. In one volume. 5s. 1915. (Ginn.)

This work is intended to replace the Wentworth *Trigonometry* which has been a standard text in America for the last generation. The principal characteristics are that the practical use of every new feature is clearly set forth before the abstract theory is developed, and that the scope is gradually enlarged as the growing ideas of the student develop greater capacity and a

natural desire for further progress. By the inclusion of a full set of five-place tables the book is made complete in itself: a shorter set of four-place tables is also given.

The whole work is excellent, in matter, style, type, and illustration; while the price is extremely moderate for the size. There is an abundance of problems and other applications, which cannot fail to inspire real interest in the subject, provided the student has sufficient time to do a good proportion of them.

Circular measure is, perhaps unfortunately for the engineer and physicist, deferred to near the end, as are also the inverse notation and graphs of the functions.

The chapter on equations and identities is especially well done. One would like to see the chapter on Applications of Trigonometry to Algebra, including De Moivre's Theorem and Exponential values of the sine and cosine, meet with the same full and careful treatment.

With regard to the tables, although the authors say they have used every practical device to avoid eye-strain, personally we think the figure type might be improved. A feature of the tables is the addition of tables for the decimal division of the degree, which the authors prophesy (rightly, we think) will in due time supplant the sexagesimal system.

Apart from the success which, due to its origin, is assured in America, this book should find a considerable sale elsewhere.

Coordinate Geometry, an Elementary Course. By PERCY COLMAN, M.A. Pp. 232. 4s. 6d. 1915. (Clarendon Press.)

The author states in his preface that this text-book is not written primarily for students who are reading or intend to read for Mathematical Honours, . . . that no great power of manipulation (in Algebra and Trigonometry) is needed, . . . that the book covers a satisfactory first year's course (presupposing a good working knowledge of graphs).

Although, from cover to cover, the book is full of good things, well put, and though the main idea is to show how Coordinate Geometry may be used as a 'useful tool' alternative to Pure Geometrical reasoning, yet this text-book seems altogether too ambitious in style, at any rate for the average student fed on the average graphs. I cannot see how the author can claim that no great power of manipulation is needed, in view of pp. 53, 78, 99, in the earlier chapters; remembering, of course, that this is a book for beginners and that the whole is supposed to be a first year's course. In my experience, all one can do in one year is to instil into a student reading for Intermediate Science examinations a really competent knowledge of the right line and circle.

Apart from this, it is surely wrong to defer oblique axes to Chapter IX. and to include Polar Coordinates in Chapter I. Since the idea of the work is to found theory on practical graphs, Polar Coordinates should start with a chapter on Polar Graphs and be more or less complete in itself; while oblique axes should be introduced immediately, it being pointed out that much of the work is true for both kinds of Cartesian coordinates.

Perhaps one of the finest features of the book is the way in which Geometrical applications continually intervene, in accordance with the author's belief that thus the student will obtain greater confidence in the new method, and can make a comparison between the two methods of Pure and Coordinate Geometry. One unfortunately rarely meets with a student, even among those reading for Mathematical Honours, who dreams of using a mixed (half pure, half coordinate) solution, or thinks of shortening his proofs by the "guess and verify" method, such as showing that $\frac{1}{3}(x_1+x_2+x_3)$, $\frac{1}{3}(y_1+y_2+y_3)$ satisfy the equations of the medians instead of the usual long-winded method of finding the coordinates of the intersection of a pair of medians, which involves a good deal of algebraical manipulation.

Another good feature is the introduction of parametric representation, though this deserves fuller treatment.

The chapter on the general equation seems to be all that can be desired, especially with regard to the careful manner in which the equation of the tangent is handled: whilst the novel treatment of the rectangular hyperbola, referred to its asymptotes with the parabola, as types of $y=ax^n$, has much in its favour.

The book concludes, as every good first-year text-book *ought*, with a short introduction to Solid Geometry.

Every teacher should at least see this book before deciding on a text-book for his students.

Numerical Trigonometry. By N. J. CHIGNELL, B.A. Pp. 126. 2s. 6d. 1915. (Clarendon Press.)

In this little introduction to Trigonometry the great feature is the careful stress laid on the idea of the trigonometrical ratios as operators. Thus a student is taught early to write down the lengths of the sides of a right-angled triangle as the product of a side and the appropriate ratio. It is a pity that the sine is defined as PN/AP instead of NP/AP; for although this is only a "numerical" Trigonometry the student should have nothing to unlearn. The problems are numerous and good, and a chapter on Addition Formulae proved by projection is added. A set of twenty test papers conclude a book for beginners that has much to recommend it.

Essays on Mathematical Education. By G. ST. L. CARSON. Pp. 139. 3s. 1914. (Ginn.)

Not only those interested in Mathematics, but every educationalist, will agree with Prof. Eugene Smith's remarks, in the introduction to this collection of essays, on the eminent qualifications of the author, which give to the collection a peculiar authoritative value.

Members of the Mathematical Association and readers of the *Gazette* will be familiar with most of these essays, but here they have an opportunity of reading and comparing the component parts of an ordered survey of the whole of elementary mathematical education; and this opportunity should on no account be missed by any one who really has the interests of education at heart.

In the first essay, Mr. Carson discusses the principles of mathematical education; he explodes the heresy that children cannot appreciate or use logical argument, but he remarks that the goal must be comprehensible, *and not a result as easily perceived as the assumption on which the truth is based*; he also points out the correct aim of illustration and verification, namely, that of inspiring confidence in the logical deductions, and not as a substitute for them.

The next essay, on Intuition, though some of the points raised admit of argument, is so immensely important that all I shall say about it is that it should be printed as an accompaniment to the Board of Education's pamphlet on the Teaching of Geometry. The latter may then fail to be as harmful, in the hands of the young inexperienced teacher, for whom supposedly it is intended, as the Board's previous pamphlet was when it advocated "interest": the result of that pamphlet, as evidenced by the text-books published, being that, instead of the pupil being made interested in the subject, the subject was debased to be made interesting to the pupil, and simply for want of a detailed definition of interest such as this essay gives of intuition.

A similar service would be performed for those who have been led to teach Practical Mathematics in stereotyped fashion, if the remarks on the distinction between "realities" and "utilities" in the next essay were read, re-read, and again re-read.

In the more detailed didactics which follow, there is hardly a point raised which is not of considerable interest and importance: *e.g.* the use of familiar terminology, the early inculcation of limits of error, the logical introduction of Algebra as generalised arithmetic, the teaching of Fractions in Arithmetic as whole numbers (of smaller units) with the abstract theory involved in "multiplication" and "division" relegated to its correct place, Algebra: and many other such matters. These are all treated in a masterly manner.

There is one passage in the last essay that is, to me at any rate, objectionable, and exemplifies my distrust of the Intuition principle in the hands of pedagogic beginners.

"Show him a triangle ABC cut out in cardboard, and make another by taking a tracing of the corner A and producing its sides until they are equal to AB and AC; and then *ask him if these triangles are an exact fit.* He will

not have much doubt about this, nor will he question the quality of all right angles, if *it be similarly suggested to him.* Next draw a circle, rule a diameter, and *ask him if one part of the curve so divided will fit the other.*"

We all occasionally ask questions of this type, but one would not expect such from Mr. Carson, especially after what he says in his essay on Intuition.

The sections on Mechanics are opportune, for in Evening Classes, at any rate, Theoretical Mechanics bids fair to be wiped out completely by a truly disastrous and accursed wave of utilitarianism.

There is a misprint in line 8 of the table on p. 118: the last "true" should be "false."

Plane Geometry, Part I. By G. St. L. Carson and David Eugene Smith. Pp. 264. 2s. 6d. (Ginn.)

The first 90 pages of this book introduce in a practical manner correct notions of position, shape and size, symmetry, similarity and the forms of plane figures and the simpler solids. Before reading *this* section of *this* book, I was thoroughly convinced by my own experience for the last fifteen years that this "playing about with ruler and compass" was sheer waste of time. I still think the same, but am not quite so certain.

Of the simple constructions on page 33, given because they are necessary for immediate use, the first three are truly intuitional, but this can hardly be said of the second three, the bisection of a line, making an angle equal to a given angle, and bisecting an angle. Nor have I found that "checking by folding" is satisfactory. Geometrical pattern drawing serves little purpose, beginners invariably drawing them badly or being too engrossed with their drawing to think of anything else. On the other hand, *Simple designing*, preferably with tinting either in water colour or crayon, is a valuable training later in "practical geometry." The revision problems on this section, forty in number, are extremely good, and boys would "work overtime" from sheer interest to get them out.* I cannot help thinking, however, that it is far preferable to build up the pupil's ideas as one goes along (that is, those he does not already possess), than to jumble them all together to begin with. Surely there is not much difference between this and the old-fashioned way of learning by rote about 50 definitions, etc. (was it ?), before one could cut off an inch-length from a given straight line, only that it is done in a much more interesting manner. This may tend to "interest" the pupil, but is not the pupil thereby rendered liable to have a distaste for Geometry when the interesting things are of the past ? Feed a child on pap too long, and you produce chronic indigestion. The only remedy is to start the teaching of Geometry at least one or even two years sooner than usual, when there could be nothing better, sounder, or more calculated to inspire, than the preliminary section of this book.

The second section opens with the necessary formal definitions of things that are now already *perfectly* known to the pupil, though in his own terminology, and no objection can be raised to such a collection of definitions as was raised to those at the beginning of Euclid.

I cannot say that I care for the style in which the proofs are written out, with reasons and explanations interpolated in small italics, as well as side references; but this must not be considered a drawback, since any teacher using the book can insist on his own method of "writing out."

The proof of the properties of parallels is, rightly or wrongly, made to depend on a postulate that a line perpendicular to one of a pair of parallels is perpendicular to the other, and conversely two straight lines both perpendicular to a straight line are parallel to one another. These are, however, fully explained, and whatever may be said, this method provides proofs for the theorems, which are neat, short, and easy of assimilation.

The proofs of the fundamental loci, the right bisector, and the bisector of an angle are rendered unnecessarily long, without increase of rigour, by the adoption by the authors of the principle of showing that no point outside the

* As corroborative evidence, I set one of my youngsters, a Boy Scout, to do one of them. He did it, and wanted to know if he could have the book, after I had finished with it, to do the rest of the "puzzles."

locus obeys the law, instead of that any point which obeys the law is on the locus.

Some of the theorems of Euclid II., viz. II., I., IV., VII., *with their algebraical equivalents*, come directly at the beginning of the authors' Book II., before the equivalent to Euclid I. 35. This is a very neat sequence, but not one that will appeal to many at first sight. It has distinct merits, however, when looked at as giving clarity of ideas with regard to area; *e.g.* the area of a triangle is considered universally as half a rectangle, and hence clear ideas of areas of rectangles are fundamental.

The exercises are universally good and sufficient in number for classes in which the alternate-question plan is not used: there is also a revision set at the end of the book.

Bill's School and Mine. By W. S. FRANKLIN. n.p. 1914. (Franklin, Macnutt & Charles, South Bethlehem, Pennsylvania.)

There is a great deal of sound common-sense underlying the apparently light tone of this little collection of essays on Education. One remark is of especial interest to organisers of technical training of the young people of England, viz. (p. 60): "The greatest educational problem of our time is to make use of commercial and industrial establishments as schools *to the extent that they are schools*": and lower down on the page, "the most important thing is that the discipline of work come again to its own in our entire system of education."

Exercises in Arithmetic and Mensuration. By P. ABBOTT, M.A. Pp. 524 +86. Answers. 4s. 6d. 1915. (Longmans.)

These exercises cover an ordinary School Course of Arithmetic, and also include elementary Mensuration, which, as the author says in his preface, must now be considered as forming an essential part of every syllabus of Arithmetic.

Part I. treats of easy fractions, decimals, weights and measures, and measurement of areas, volumes, capacity and weight. The whole is very skilfully handled; but one is inclined to wonder how it is all to come in. There are in this part about 700–800 questions, many of them demanding preliminary model making or drawing on squared paper and counting, access to a decent balance and other materials. In addition, there are nearly 100 examples for revision. Some of the questions are really physics laboratory experiments: otherwise one becomes rather anxious about a cubic inch of water contained in a model box made of squared paper (which never could measure a cubic inch with any degree of accuracy worth speaking about), even if rendered watertight by the use of paraffin wax. The use of the alternative, sand, is worse, for the average boy, as I used to know him, would take particular care that, if any of this were spilled, it should find a resting place anywhere but in the proper place: lead shot also are a nuisance, and may easily cause a fall or a twisted ankle. Apart from this, one feels one would like to be a boy of 8–9 once more and to start in this pleasant manner; then comes the thought—Should we be any better off then than now? Should we have been trained to last out through the uninteresting grind that some of our present work must be, no matter how much our hearts are in it?

Part II. completes the treatment of fractions with the same interesting wealth of illustration from mensuration and other sources when available. On page 179 there seems to be a misprint in (3), which should read: The signs of *multiplication* and division, etc.; also in (4) it surely would be preferable to use the usual "definition" of "*of*," viz. that it is equivalent to $\times$ together with a bracket; for "quantities" connected by $\times$, $\div$ are also to be considered as forming one "quantity." The treatment of fractions would be more sound if, according to Mr. Carson's method, the idea of "multiplication" and "division" were left till later, such questions as $7\frac{3}{4} \div 2\frac{1}{2}$ being set as concrete problems. If such problems cannot be concocted, then there is no necessity to introduce such questions at all until multiplication and division by fractions are introduced and logically defined in Algebra.

Part III. deals with ratio and proportion in the same inspiring manner, and we have illustration by means of gradients and projections of areas,

the straight line law, and drawing to scale. One, however, is rather staggered to find, when only half-way through an Arithmetic, that one is expected to be able to follow the proof of the formula for the volume of a frustum of a cone or of the area of a zone on a sphere.

Part IV. starts with contracted methods. This is the only serious fault I can find with the book. *This should have been in the first book.* Mr. Abbott had the chance of a generation to show England how Arithmetic should be taught. He starts with measurement, *essentially approximate*, and—the inference is obvious—left to right multiplication, upper and lower limits,* and his last chapter examples on practical and squared-paper work on gradients and rates of increase. Had this been done, I am confident that in ten years time you would enter few schools in England without seeing an Abbott on the desks; and mathematical teachers would owe him a debt hard to repay.

J. M. CHILD.

CORRESPONDENCE.

THE EDITOR OF THE *Mathematical Gazette.*

DEAR SIR,—I have read with much interest Mr. Lodge's article in No. 116 of the *Gazette*, pp. 39-41. In an article published in 1907, in the *Annaes da Academia Polytechnica do Porto*, vol. 2, I proposed the following method for calculating the radius of curvature ρ at any point Ω of any curve C. Let ΩA, ΩB, the tangent and normal respectively at the point Ω, be taken as coordinate axes, and let ξ, η be the coordinates with respect to these lines of a point close to Ω (on one of the branches meeting at Ω); we then have

$$\rho = \mathrm{L}_{\xi=0} \frac{\xi^2}{2\eta}.$$

I showed how easily the method may be applied, giving as examples the folium of Descartes, the strophoid, the conchoid of Sluse, Maclaurin's trisectrix, the conchoid of Nicomedes, Pascal's limaçon, and all the Rhodoneae. My results were also published in the second edition of my *Specielle algebraische und transcendente ebene Kurven.*

This letter is not written for the purpose of claiming priority, but with the object of drawing the attention of the readers of the *Gazette* to a simple and general method which, if I am not mistaken, may often be found useful.

—Yours sincerely,

GINO LORIA.

Genoa, April 14, 1915.

DEAR SIR,—Dr. T. J. I'a Bromwich has kindly pointed out to me that my note on "Singular solutions of differential equations of the second order," in the December number of the *Mathematical Gazette*, is seriously in error.

Firstly, the condition $$\frac{\partial F}{\partial x} \cdot \frac{\partial^2 F}{\partial y \partial t} = \frac{\partial F}{\partial y} \cdot \frac{\partial^2 F}{\partial x \partial t}$$

is the condition for super-osculation (not osculation) as a rule. The proof of this statement is given in the *Quarterly Journal*, 34 (1902), p. 98, by Bromwich and Hudson. The process I suggested gives singular solutions, but the proof needs modification.

Secondly, the statement that curves "1 and 2 meet at P, which is consecutive to Q and R," is not necessarily true; and is, for example, not true in Ex. 2 of my paper, as Dr. Bromwich points out. In fact, the use of the word "consecutive" is always fraught with danger!—Yours faithfully,

HAROLD HILTON.

* See Carson, *Mathematical Education*, page 53 *et seq.*

THE HAPPY LAND.*

I HEAR them speak of a Happy Land,
The chosen researchers a joyful band,†
Professor, ah where is that radiant shore,
Shall I attain it and weep no more ?
Take care, take care, my child.

Is it where the Geometer draws his base,
And elegant quadrics float through space,
Where the circular points are the open door,
And conics osculate evermore ?
Not there, not there, my child.

Does it lie 'mid Algebra's stern array,
Where the Law of Symmetry points the way,
And the path leads up through ascending powers
To the hilltop won after weary hours ?
Not there, not there, my child.

Is it set in the space of the Dead-alive,
Where the Non-Euclidean seems to thrive,
Where nothing is ever the form it seems,
And the Absolute haunts in ghostly dreams ?
Not there, not there, my child.

It must be then in that region fair
Where the Calculus scents the fragrant air,
And the infinitesimal's gifts combine,
Giving man a mastery half divine ?
Not there, not there, my child.

Can it lie, perchance, in the active sphere
Of the highly technical engineer,
Who silent stands in the foremost place,
Hewing the path for the human race ?
Not there, not there, my child.

It lies afar on the Z' plane,
Conformal, mapped by a Cauchy brain,
Where genius sees with the complex i,
And the Spirit of Functions dwells thereby.
It is there, it is there, my child.

ANON.

THE LIBRARY.

CHANGE OF ADDRESS.

THE Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian acknowledges, with thanks, *Co-ordinate Geometry*, by Percy Coleman, and *Numerical Trigonometry*, by N. J. Chignell, both given by the Clarendon Press.

SCARCE BACK NUMBERS.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them :

Gazette No. 8 (very important).
A.I.G.T. Report, No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

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* Reprinted by kind permission of the author, who prefers to remain anonymous.
† The Research School of the Mathematical Department at Edinburgh University.

BOOKS, ETC., RECEIVED.

Numerical Trigonometry. By N. J. CHIGNELL. Pp. 124+xii. 2s. 6d. (with or without Answers). 1914. (Clarendon Press.)

Coordinate Geometry. An Elementary Course. By P. COLEMAN. Pp. 240. 4s. 6d. 1914. (Clarendon Press.)

Usefulness of the Sigma Xi. By G. A. MILLER. Pp. 67-70. (*Sigma Xi Quarterly*, Vol. II. No. 3. September, 1914.)

Recent Mathematical Activities. By G. A. MILLER. (*Popular Science Monthly* November, 1914. Pp. 457-462.)

American Journal of Mathematics. Edited by F. MORLEY. Vol. XXXVII. No. 1. Jan. 1915. 5$ per vol.

Generalisations of Geodesic Curvature and a Theorem of Gauss concerning Geodesic Triangles. G. A. BLISS. *On the Medians of a Closed Convex Polygon.* ARNOLD EMCH. *The Rational Sextic Curve, and the Cayley Symmetroid.* J. R. CONNER. *Limited and Illimited Linear Difference Equations of the Second Order with Periodic Coefficients.* T. FORT. *Finite Groups of Plane Birational Transformations with eight Fundamental Points.* F. R. SHARPE. *Conics through Inflections of Self-Projective Quartics.* F. R. SHARPE. *Concerning an Analogy between Formal Modular Invariants and the Class of Algebraical Invariants Called Booleans.* O. E. GLENN. *Periodic Orbits on a Smooth Surface.* D. BUCHANAN. *On Foucault's Pendulum.* W. D. MACMILLAN.

Norman's Arithmetic for Schools. By J. S. NORMAN and F. K. NORMAN. Pp. xvi+278. 2s. With Answers, 2s. 6d. 1915. (The Year Book Press.)

Ecuaciones integrales lineales. By H. BROGGI. Pp. 36.

Determinación de H, Componente Horizontale de la Intensidad del Campo Magnético terrestre en La Plata. By T. ISNARDI. Pp. 47-65.

El Paramagnetismo en Función de la Temperatura y de la Densidad. By R. GAUS. Pp. 71-80. (Reprints from Vol. I. Ser. Matem., and Part II. Vol. I. Serie Fisica. Contribución al Estudio de las Ciencias físicas y matemáticas.)

Mathematical Papers for Admission into the Royal Military Academy and the Royal Military College. 1905-1914. With Answers. Edited by R. M. MILNE, M.A. 6s. 1915. Sept.-Nov. 1914. Pp. 28. 1s. net. 1915. (Macmillan.)

Electromagnetic Waves in a perfectly conducting Tube. By L. SILBERSTEIN. Reprinted from Proc. Roy. Soc. A. Vol. 91. Pp. 170-179. 1915.

The American Mathematical Monthly. Vol. XXI. 1914. Vol. XXII. Nos. 1, 2. Edited by T. E. SLAUGHT and others. 2$ per ann. Lancaster, Pa. U.S.A.

A Treatise on the Analytic Geometry of Three Dimensions. By G. SALMON. Fifth Edition. Vol. II. Edited by R. A. P. ROGERS. Pp. xvi+334. 7s. 6d. net. 1915. (Longmans, Green.)

Plane Trigonometry. By H. S. CARSLAW. Pp. xviii+293+xi. 4s. 6d. 1915. (Macmillan.)

Solutions of the Questions in Plane Trigonometry. By H. S. CARSLAW. Pp. 179. 6s. 6d. net. 1915. (Macmillan.)

Junior Algebra. By A. G. CRACKNELL and A. BARRACLOUGH. Pp. vi+280. Without Answers, 2s. 6d. With Answers, 3s. 1915. (Univ. Tutorial Press.)

Calculus Made Easy; being a very simplest Introduction to those beautiful Methods of Reckoning which are generally called by the terrifying Names of the Differential Calculus and the Integral Calculus. By F.R.S. Second edition, enlarged. Pp. x+265. 2s. net. 1914. (Macmillan.)

Homogeneous Linear Substitutions. By H. HILTON. Pp. viii+184. 12s. 6d. net. 1914. (Clarendon Press.)

The Theory of Proportion. By M. J. M. HILL. Pp. xx+108. 8s. 6d. net. 1914. (Constable.)

The Laws of Algebra. An Elementary Course in Algebraic Theory. By A. G. CRACKNELL. Pp. vi+68. 1s. 1915. (Univ. Tutorial Press.)

Mathematical Problem Papers. Compiled and arranged by the REV. E. M. RADFORD. 2nd Edition. Pp. 204. 4s. 6d. net. 1915. (Cam. Univ. Press.) *Solutions* to above. Pp. vi+560. 10s. 6d. net. 1915. (Cam. Univ. Press.)

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